Answers for NSSH exam paper 2 type of questions, based on the syllabus part 2 (includes 16)

Section 1 Integration

1. (a) Integrate with respect to x: $\int \frac{dy}{dx}\,dx = \int (\frac{6}{x^2} + 2)dx$ or $y = -\frac{6}{x} + 2x + c$

The curve passes through $P(3,10)$ so $10 = -6/3 + 2 \times 3 + c$ this means $c = 6$ so that

the required equation is: $y = -\frac{6}{x} + 2x + 6$

(b) Substitute $x = 3$ in: $\frac{dy}{dx} = \frac{6}{x^2} + 2 \rightarrow$ the derivative is then $6 \div 9 + 2 = 2^2/_3$

So the gradient of the tangent is $2^2/_3$, the equation becomes $y = 2^2/_3\, x + c$. $P(3, 10)$ is on the line so $10 = 2^2/_3 \times 3 + c$ this means $c = 2$. The equation of the tangent is: $\underline{y = 2^2/_3\, x + 2}$

2. $\int\left(\frac{2}{x}-\frac{4}{x^3}\right)dx = 2\int\left(\frac{1}{x}-\frac{2}{x^3}\right)dx = 2[\ \ln x + \frac{1}{x^2}\ ] + c = 2\ln x + \frac{2}{x^2} + c$

3. Use the substitution $p = 3 - x$ then $\frac{dp}{dx} = -1$ so $dp = -dx$ so that $\int \frac{2}{\sqrt{3-x}}dx = 2\int \frac{-1}{\sqrt{p}}\,dp =$

$-2 \times 2 \times \sqrt{p} + c = -4\sqrt{p} + c = -4\sqrt{3-x} + c.$

4. Integrate : $\frac{dy}{dx} = -x + 2$ with respect to x: $\int \frac{dy}{dx}dx = \int(-x+2)dx$ or $y = -½\, x^2 + 2x + c$

(2, –1) is on the curve so $-1 = -2 + 4 + c$ so that $c = -3$. The equation of the curve is then: $\underline{y = -½\, x^2 + 2x - 3.}$

5. (a) Use the substitution $p = 2x - 3$ then $\frac{dp}{dx} = 2$ so that $dp = 2dx$ this means $\int \frac{1}{2x-3}dx = \frac{1}{2}\int \frac{2}{2x-3}dx =$

$\frac{1}{2}\int \frac{1}{p}dp = \frac{1}{2}\ln p + c$ substitute $2x - 3$ for p and you get: $\underline{½ \ln(2x - 3) + c}$ as an answer.

(b) $\int_0^2 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_0^2 = \frac{1}{2}e^4 - \frac{1}{2}$

6. (b) (i) $\int e^{2x-3}dx$ use a substitution $u = 2x - 3 \rightarrow \frac{du}{dx} = 2$ so that $2dx = du$ or $dx = ½\, du$

$$\int e^{2x-3}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{2x-3}$$

(ii) $\int_0^3 \frac{1}{2x+3}\,dx$ use a substitution $u = 2x + 3 \rightarrow \frac{du}{dx} = 2$ so that $2dx = du$ or $dx = ½\, du$

$$\int_0^3 \frac{1}{2x+3}\,dx = \frac{1}{2}\int_3^9 \frac{1}{u}\,du = \frac{1}{2}\left[\ln u\right]_3^9 = \frac{1}{2}\{\ln 9 - \ln 3\} = 0.5493$$

7 **(b)** Find $\int \frac{2}{3x-1}\,dx = 2\int \frac{1}{3x-1}\,dx = \frac{2}{3}\ln(3x-1)$

8. $\int\left(\frac{2}{x}-\frac{4}{x^2}\right)dx = 2\int\frac{1}{x}dx - 4\int x^{-2}dx = 2\ln x + 4x^{-1} + c$

9. (i) $\int\frac{1}{2x+3}dx$ = ½ ln (2x + 3) + c. This can be solved as well solved with a substitution:

Let $p = 2x + 3$ so $\frac{dp}{dx} = 2$ so that ½$dp = dx$. Then $\int\frac{1}{2x+3}dx = \frac{1}{2}\int\frac{1}{p}\,dp = \frac{1}{2}\ln p + c = \frac{1}{2}\ln(2x+3) + c.$ =

(ii) $\int_0^2 e^{3x-2}dx = \left[\frac{1}{3}e^{3x-2}\right]_0^2 = \frac{1}{3}\left\{e^4 - e^{-2}\right\}$ [This is an exact answer so it can be left like this; the approximate value is 18.2]

Section 2 Differentiation and finding gradients

1. First find the x coordinate when y = ln 8: ln 8 = 5 ln x – ln 4 or ln 8 + ln 4 = 5 ln x using the first law of logs we get: ln 32 = 5 ln x or ln 32 = ln x^5 [Third law of logs] so that x^5 = 32. This means x = 2.

Differentiate the curve with respect to x: $\frac{d}{dx}(5\ln x - \ln 4) = \frac{5}{x}$**.** Substitute x = 2, gives a gradient 2.5

2. $\frac{d}{dx}e^{x^2+1} = 2x\,e^{x^2+1}$

3. (a) $f'(x) = 2x + \frac{1}{x^2}$ stationary point → $f'(x) = 0$ so $2x + \frac{1}{x^2} = 0$ (multiply with x^2) → $2x^3 + 1 = 0$ or $x^3 = -½$

$x = -\sqrt[3]{0.5} \approx -0.794$ the y coordinate of the stationary point is $f(-\sqrt[3]{0.5}) = [(-\sqrt[3]{0.5})]^2 - \frac{1}{\left(-\sqrt[3]{0.5}\right)} = 1.89$

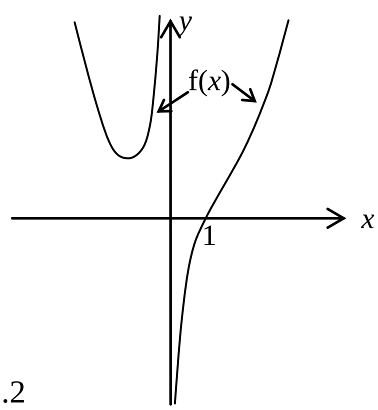
So the coordinates of the stationary point is (–0.794, 1.89)

$f''(x) = 2 - \frac{2}{x^3} > 0$ for $x < 0$ so the stationary point is a minimum.

(b) see **(a).**

(c) $f'(x) = 2x + \frac{1}{x^2}$ both terms are positive for $x > 0$ so all gradients of the curve are positive in the first and fourth quadrant.

(d)



4. **(a)** $\frac{d}{dx}\left[\frac{\sqrt[5]{x^3}}{3} + 10\right] = \frac{d}{dx}\left[\frac{1}{3}x^{\frac{3}{5}} + 10\right] = \frac{1}{3}\times\frac{3}{5}\,x^{-0.4} = 0.2x^{-0.4} = \frac{0.2}{\sqrt[5]{x^2}}$

(b) $\frac{d}{dx}\left[\frac{3x+2}{x^2}\right] = \frac{d}{dx}\left[\frac{3}{x} + \frac{2}{x^2}\right] = -\frac{3}{x^2} - \frac{4}{x^3}$

5. If $y = 2x^2 - 6x + 20$ *) then $\frac{d}{dx}[2x^2 - 6x + 20] = 4x - 6$. This equation gives the value of the gradients of the tangents to the curve. The tangent has gradient 2 so solve $4x - 6 = 2$ or $x = 2$ substitute this in *):

$2 \times 4 - 12 + 20 = 16$ so the point of touch is (2, 16). This is as well the point the tangent is passing through. So substitute (2, 16) in the tangent: $16 = 2 \times 2 + c$ so $\underline{c = 12.}$

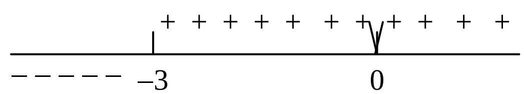
6. **(a)** $\frac{d}{dx}\ln x^3 = \frac{3x^2}{x^3} = \frac{3}{x}$

(b) $\frac{d}{dx}\ \ln 3x = \frac{3}{3x} = \frac{1}{x}$

7. **(a)** $f'(x) = 4x^3 + 12x^2 = 4x^2(x + 3)$

(b) $f'(x) = 0 \rightarrow 4x^3 + 12x^2 = 0$ or $4x^2(x + 3) = 0$ so $x = 0$ or $x = -3$ so the stationary points are (0, 0) and (–3, – 27)

(c) Solution I A sign diagram of $f'(x)$ looks like:



So (–3, – 27) is a minimum and (0,0) is a point of inflection.

Solution II First determine $f''(x) = 12x^2 + 24x = 12x(x + 2)$ with $f''(-3) > 0$ so (–3, – 27) is a minimum.
As well: $f''(0) = 0$ so (0, 0) is a point of inflection.

8. **(a)** $\frac{d}{dx}\ln 2x = \frac{2}{2x} = \frac{1}{x}$

9. **(a)** let $y = \frac{\sqrt[5]{x^2}}{3} + 10 = \frac{1}{3}x^{0.4} + 10$ then $\frac{dy}{dx} = \frac{2}{15}x^{-0.6}$

(b) let $y = \frac{2x^3 - 5x}{x^2} = 2x - 5x^{-1}$ then $\frac{dy}{dx} = 2 + 5x^{-2}$

10. **(a)** Let $y = e^{x^2-3}$ then $y' = 2xe^{x^2-3}$.

11. **(a)** $\frac{dy}{dx}\ \ln(2x^2 + 3) = \frac{4x}{2x^2+3}$

f(x)=x^3+3x^2+4x+4

(b) $\int \frac{x}{2x^2+3}dx = 0.25\int \frac{4x}{2x^2+3}dx = 0.25\ln(2x^2+3) + C$

12. **(a)** $f'(x) = 3x^2 + 6x + 4$
Try to solve $3x^2 + 6x + 4 = 0$…* the discriminant is:
$36 - 4 \times 3 \times 4 = -12$ so the equation * has no solution.

(b) Point of inflexion ➔ $f''(x) = 6x + 6 = 0$ so $x = -1$ so the point of inflection is (–1, f(–1)) = (–1, 2).

(c) Use the table function on your calculator. Point of intersection with the x-axis: (–2, 0) and the y-axis (0, 4).

(d) See graph.

13. $\frac{d}{dx}e^{x^2} = 2x\ e^{x^2}$

Section 3 Equations (including trig equations and identities)

1.(a) Divide $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$, it becomes: $\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$ or $\tan^2 x + 1 = \sec^2 x$

Substitute this in the given equation: $2(\tan^2 x + 1) + \tan x - 5 = 0$ so the equation is: $\underline{2\tan^2 x + \tan x - 3 = 0}$

(b) Let $\tan x = p$ then the quadratic equation will become: $2p^2 + p - 3 = 0$ find two numbers product –6; sum 1.
These are 3 and –2 so the eq. changes to $2p^2 + 3p - 2p - 3 = 0$ or $p(2p + 3) - 1(2p + 3) = 0$ or $(p - 1)(2p + 3) = 0$
so $p = 1$ or $\tan x = 1$ which means $x = \underline{0.25\pi \text{ rad} = 0.785 \text{ rad}}$ or $\underline{1.25\pi \text{ rad} = 3.93 \text{ rad}}$
or $p = -1.5$ or $\tan x = -1.5$ $\tan^{-1}(-1.5) = -0.98279$ so $x = (\pi - 0.98279)$ rad $\underline{= 2.16 \text{ rad}}$ or $x = 2\pi - 0.98279 =$ $\underline{5.30 \text{ rad}}$

2. **(a) (i)** $\frac{\sin x + \cot x \cos x}{\sin x} \equiv \operatorname{cosec}^2 x$

LHS = $\frac{\sin x + \frac{\cos x}{\sin x}\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x}$ after multiplying numerator and denominator with $\sin x$.

Using $\sin^2 x + \cos^2 x = 1$ we get: LHS = $\frac{1}{\sin^2 x}$ which is indeed equal to the RHS. Q.e.d.

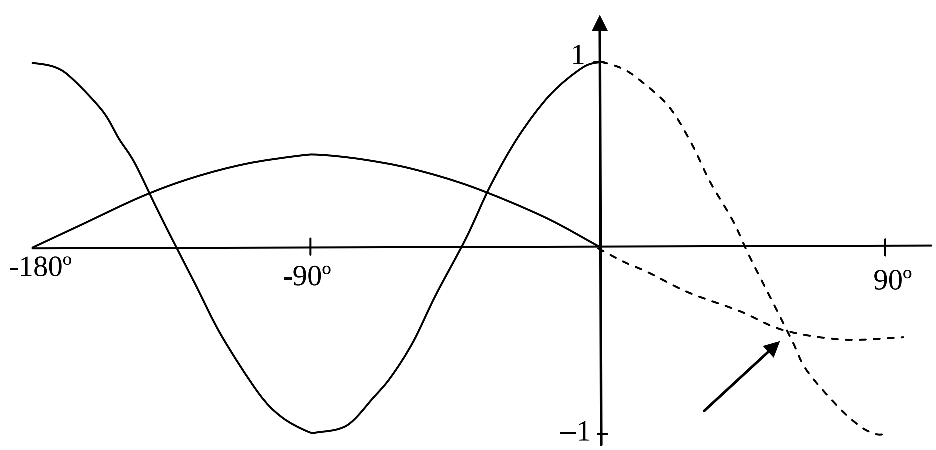
(ii) For $x = 0^\circ$ (the denominator is zero)

(b) (i) Divide $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$, it becomes: $\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$ or $\tan^2 x + 1 = \sec^2 x$

Make $\tan^2 x$ subject: $\tan^2 x = \sec^2 x - 1$ and substitute in: $2\tan^2\theta - 3\sec\theta = 0$; you get:
$2(\sec^2 x - 1) - 3\sec\theta = 0$ or $2\sec^2 x - 3\sec x - 2 = 0$.

(ii) Let $\sec x = p$ the equation changes to
$2p^2 - 3p - 2 = 0$ factorize: $2p^2 - 4p + p - 2 = 0 \rightarrow 2p(p - 2) + 1(p - 2) = 0$ or $(p - 2)(2p + 1) = 0$.
① This gives solutions $p = 2$ or $\sec x = 2$ or $1/\cos x = 2$ this means $\cos x = ½$ so $x = \frac{1}{3}\pi$ or $x = 1\frac{2}{3}\pi$.
② As well: $p = -½$ or $\sec x = -½$ or $1/\cos x = -½$ this means $\cos x = -2$ so no further solution here.

(c) (i)



(ii) Extend the graphs a bit, so there is one solution of the equation $f(x) = g(x)$ in the interval $0^\circ \le x \le 90^\circ$.

3. **(a)** $\frac{\sin x}{1 - \cos x} - \frac{\cot x}{1} \equiv \operatorname{cosec} x$

LHS = $\frac{\sin x(1+\cos x)}{1-\cos^2 x} - \frac{\frac{\cos x}{\sin x}}{1} = \frac{\sin x(1+\cos x)}{\sin^2 x} - \frac{\cos x}{\sin x} = \frac{1+\cos x}{\sin x} - \frac{\cos x}{\sin x} = \frac{1}{\sin x} = \operatorname{cosec} x$ q.e.d.

(b) $\cot \frac{1}{2}\theta = -2.987 \rightarrow \tan \frac{1}{2}\theta = 1/-2.987 = -0.3347840\ldots$ so that $\frac{1}{2}\theta = -18.509724\ldots + 180^\circ k;\ k \in \mathbb{Z}$

This means $\theta = -37.01944984\ldots + 360^\circ k = -37^\circ + 360^\circ k$ so $\underline{\theta = 323^\circ}$ [Taking $k = 1$]

(c) (i) $5 - 7\sin x - 2\cos^2 x = 0$ *) substitute $\cos^2 x = 1 - \sin^2 x$ so *) becomes: $5 - 7\sin x - 2(1 - \sin^2 x) = 0$

Rewrite: $2\sin^2 x - 7\sin x + 3 = 0$ **)

(ii) Let $\sin x = p$ then **) becomes: $2p^2 - 7p + 3 = 0$ or $2p^2 - 6p - p + 3 = 0$ or $2p(p-3) - 1(p-3) = 0$

So that $(p-3)(2p-1) = 0$ this gives $\sin x = \frac{1}{2}$ so that $x = 30^\circ$ or $x = 150^\circ$ in radians: $x = \frac{1}{6}\pi$ radians or

$x = \frac{5}{6}\pi$ radians; $p = 3$ or $\sin x = 3$ does not give any solution.

[Remark: one can express $\frac{1}{6}\pi$ radians and $\frac{5}{6}\pi$ radians as decimals. This is not wrong but at the same not asked for. If you round off wrongly, marks can/will be subtracted.]

4. (a) $\sin 3x = -\frac{1}{2} \rightarrow 3x = \sin^{-1}(-\frac{1}{2}) = -0.5236.. + 2\pi k$ rad with $k \in \mathbb{Z}$ …..①

or $3x = -\pi + 0.5236.. + 2\pi k$ rad $k \in \mathbb{Z}$ …… ②

① gives: $x = -0.1745329.. + \frac{2}{3}\pi k$ that means $\underline{x = 1.92}$ rad ($k = 1$)

② gives: $x = -0.872664.. + \frac{2}{3}\pi k$ that means $\underline{x = 1.22 \text{ rad}}$ ($k = 1$)

(b) $\tan x - \cot x \equiv \dfrac{1 - 2\cos^2 x}{\sin x \cos x}$.

$$\text{LHS} = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{(1 - \cos^2 x) - \cos^2 x}{\sin x \cos x} = \frac{1 - 2\cos^2 x}{\sin x \cos x} \quad \text{q.e.d.}$$

5. (a) $\sec x - \dfrac{\cos x}{1 + \sin x} \equiv \tan x$

$$\text{LHS} = \frac{1}{\cos x} - \frac{\cos x(1 - \sin x)}{1 - \sin^2 x} = \frac{1}{\cos x} - \frac{\cos x(1 - \sin x)}{\cos^2 x} = \frac{1 - 1 + \sin x}{\cos x} = \tan x$$

(b) $\operatorname{cosec} 2x = 4 \rightarrow \dfrac{1}{\sin 2x} = 4 \rightarrow \sin 2x = 0.25 \rightarrow$

$2x = 0.252680.. + 2\pi k$ rad…….①

or $2x = \pi - 0.252680.. + 2\pi k$ rad…….②

① gives $x = 0.126340 + \pi k$ so $x = \underline{0.126 \text{ radians}}$

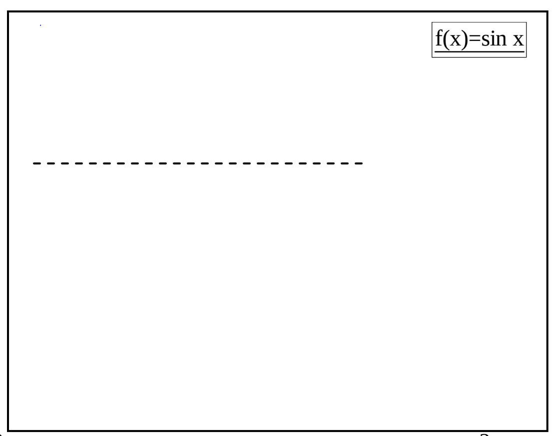
or $\underline{x = -3.02 \text{ rad}}$

② gives $2x = 2.889 + 2\pi k$ radians or

$x = 1.4442 + \pi k$

this give the solutions $\underline{x = 1.44 \text{ radians}}$ ($k = 0$)

or $\underline{x = 1.444 - \pi = -1.70 \text{ radians}}$ ($k = -1$)



(c) (i) $4 - 5\cos x - 2\sin^2 x = 0$ *) substitute $\sin^2 x = 1 - \cos^2 x$ *) becomes: $4 - 5\cos x - 2(1 - \cos^2 x) = 0$

Rewrite: $2\cos^2 x - 5\cos x + 2 = 0$.

(ii) Substitute $\cos x = p \rightarrow 2p^2 - 5p + 2 = 0$ or $2p^2 - 4p - p + 2 = 0$ or $2p(p-2) - 1(p-2) = 0$

so that $(p-2)(2p-1) = 0$ this gives $p = 2$ [No sol.] and $p = \frac{1}{2}$ or $\cos x = \frac{1}{2}$ this means $\underline{x = 60^\circ \text{ or } x = 300^\circ}$

6. (a) $\cos 3x = \frac{1}{2}$ general solution is: $3x = \frac{1}{3}\pi + 2k\pi$ ..... ① with k an integer or $3x = 1\frac{2}{3}\pi + 2k\pi$......②

① leads to $x = \frac{1}{9}\pi + \frac{2}{3}k\pi$ which will give solutions $x = -\frac{5}{9}\pi$; $\frac{1}{9}\pi$ and $\frac{7}{9}\pi$ [$k \in \mathbb{Z}$]

② leads to $x = \frac{5}{9}\pi + \frac{2}{3}k\pi$ which will give solutions $x = -\frac{7}{9}\pi$; $-\frac{1}{9}\pi$ and $\frac{5}{9}\pi$

(b) $\frac{\operatorname{cosec} x \times \tan x}{\tan x + \cot x} \equiv \sin x$ LHS = $\frac{\frac{1}{\sin x} \times \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{\left(\frac{1}{\sin x} \times \frac{\sin x}{\cos x}\right) \times \sin x \cos x}{\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) \times \sin x \cos x} = \frac{\frac{\sin x}{1}}{\frac{1}{1}} = \sin x$ = RHS.

7. (a) LHS = $\frac{\tan x + 1}{\tan x - 1} = \frac{\left(\frac{\sin x}{\cos x} + 1\right) \times \cos x}{\left(\frac{\sin x}{\cos x} - 1\right) \times \cos x} = \frac{\sin x + \cos x}{\sin x - \cos x}$

RHS = $\frac{\sec x + \operatorname{cosec} x}{\sec x - \operatorname{cosec} x} = \frac{\frac{1}{\cos x} + \frac{1}{\sin x}}{\frac{1}{\cos x} - \frac{1}{\sin x}} = \frac{\left(\frac{1}{\cos x} + \frac{1}{\sin x}\right) \times \sin x \cos x}{\left(\frac{1}{\cos x} - \frac{1}{\sin x}\right) \times \sin x \cos x} = \frac{\sin x + \cos x}{\sin x - \cos x}$

So LHS = RHS.

(b) cot $2x = 3$ or tan $2x = \frac{1}{3}$ then $2x = \tan^{-1}(\frac{1}{3}) = 18.43^\circ + k\,180^\circ$ with $k \in \mathbb{Z}$, divide by 2:

$x = 9.217^\circ + k \times 90^\circ$.

Take $k = -2$ then $x = -171^\circ$

Take $k = 0$ then $x = 9^\circ$

Take $k = 1$ then $x = 99^\circ$.

(c) Solve for x : $-2\cos^2 x - 3\sin x = 0$ ……..* for $0 \le x \le 2\pi$ radians

Use $\cos^2 x = 1 - \sin^2 x$ then * $-2(1 - \sin^2 x) - 3\sin x = 0$ rewrite: $2\sin^2 x - 3\sin x - 2 = 0$.

Rewrite: $(2\sin x + 1)(\sin x - 2) = 0$. The only acceptable solution is $\sin x = -½$.

$x = -\frac{1}{6}\pi$ radians. So the solutions are: $x = 1\frac{5}{6}\pi$ radians or $x = 1\frac{1}{6}\pi$ radians.

8. [Paper 2 2015 Q 12]

(a) (i) LHS = $\frac{\cos\theta}{1+\cos\theta} + \frac{\cos\theta}{1-\cos} = \frac{\cos\theta(1-\cos\theta) + \cos\theta(1+\cos\theta)}{1-\cos^2\theta} = \frac{2\cos\theta}{\sin^2\theta} = 2\frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} = 2\cot\theta \times \operatorname{cosec}\theta$

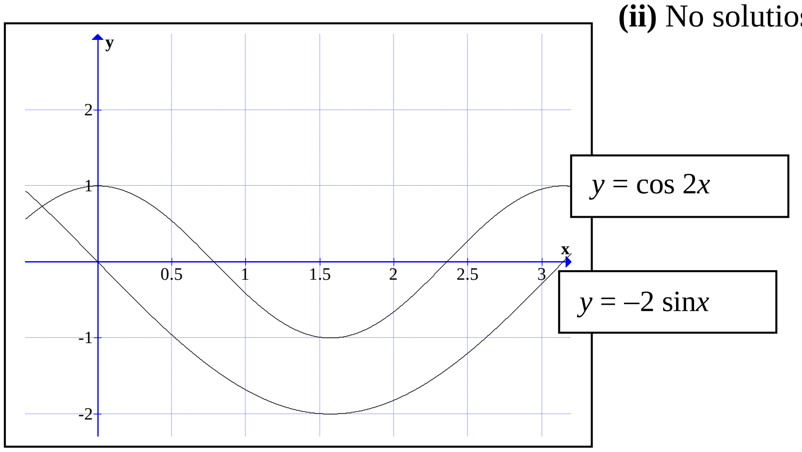
(ii) The identity is not defined when $1 - \cos\theta = 0$ for $\theta = 0^\circ$

(b) (i) $\tan^2 x - \sec x = 1$….*. Use the identity $\tan^2 x + 1 = \sec^2 x$ in eq * you get: $\sec^2 x - 1 - \sec x = 1$ or $\sec^2 x - \sec x - 2 = 0$

(ii) $\tan^2 x - \sec x = 1$ or $\sec^2 x - \sec x - 2 = 0$ which is $(\sec x - 2)(\sec x + 1) = 0$ so $\sec x = 2$…① or $\sec x = -1$….②

① can be rewritten as $\cos x = ½$ or $x = 60^\circ$ and ② as $\cos x = -1$ which has no solution in $[0^\circ, 90^\circ]$

(c) (i)



(ii) No solutios.

9. (a) $\dfrac{1}{\sec\theta - \tan\theta} \equiv \dfrac{1+\sin\theta}{\cos\theta}$

$$\text{LHS}=\frac{1}{\sec\theta-\tan\theta}=\frac{1}{\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}=\frac{1\times\cos\theta}{\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)\times\cos\theta}=\frac{\cos\theta}{1-\sin\theta}=\frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

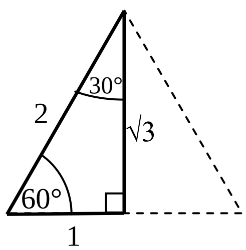
$$=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}=\frac{1+\sin\theta}{\cos\theta}=\text{RHS } qed$$

(b) $\cos 2x = -½\sqrt{3}$ so $2x = \cos^{-1}(-½\sqrt{3}) + 2k\pi$ …..① or $2x = 2\pi - \cos^{-1}(-½\sqrt{3}) + 2k\pi$….② with $k \in \mathbb{Z}$.

Divide ① by 2 gives: $x = 1.31 + k\pi$ which gives: <u>$x = 1.31$ rad</u> ($k = 0$) or <u>$x = -1.83$ rad</u> ($k = -1$)

Divide ② by 2 gives: $x = \pi - 1.31 + k\pi$ which gives: $x = 1.83$ ($k = 0$) or <u>$x = -1.31$</u> ($k = -1$).

This solution is acceptable but not very accurate. Some calculators give exact solutions in terms of π, which are preferable. Even without that calculator more exact solutions can be found using the special Δ and the graph of the cosine function.



From the Δ→ $\cos 30° = ½\sqrt{3}$ or $\cos \frac{1}{6}\pi = ½\sqrt{3}$.

Then $\cos \frac{5}{6}\pi = -½\sqrt{3}$..① or $\cos \frac{7}{6}\pi = -½\sqrt{3}$…②

① gives: $2x = \frac{5}{6}\pi + 2k\pi \rightarrow x = \frac{5}{12}\pi + k\pi$.

With solutions: $x = \frac{5}{12}\pi$ ($k = 0$) or $x = -\frac{7}{12}\pi$ ($k = -1$)

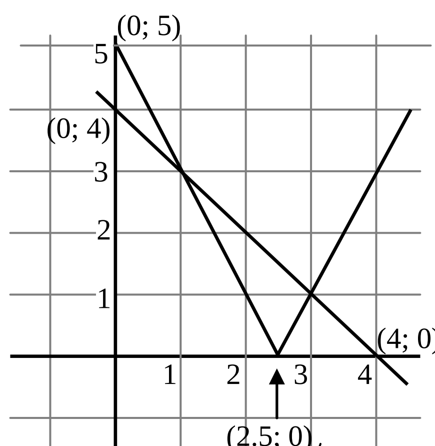
② gives: $2x = \frac{7}{6}\pi + 2k\pi \rightarrow x = \frac{7}{12}\pi + k\pi$.

With solutions: $x = \frac{7}{12}\pi$ ($k = 0$) or $x = -\frac{5}{12}\pi$ ($k = -1$).

<u>Section 4 Equations and graphs of absolute values</u>

1.

(a)



(b) $|2x-5| = 0$ for $x = 2½$

Using the definition of absolute values we get:

for $x \geq 2½ \rightarrow$ $|2x-5| + x = 4$ becomes $2x - 5 + x = 4$ or

$3x = 9$ or <u>$x = 3$.</u>

for $x < 2½ \rightarrow$ $|2x-5| + x = 4$ becomes $-2x + 5 + x = 4$ or

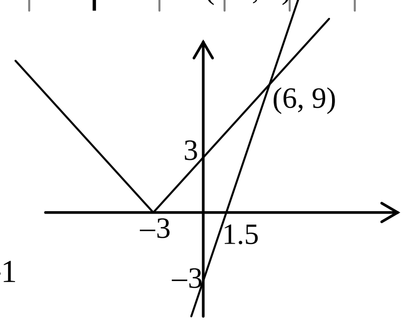
$-x = -1$ or <u>$x = 1$.</u>

2. (a) $|x + 3| = 2x - 3 \rightarrow x + 3 = 0$ for $x = -3$.

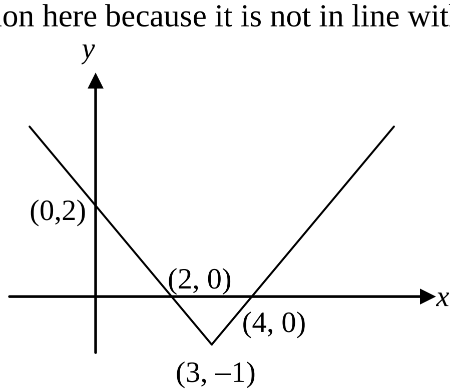
For $x \geq -3$ the equation becomes: $x + 3 = 2x - 3$ or <u>$x = 6$</u>.

For $x < -3$ the equation becomes: $-x - 3 = 2x - 3 \rightarrow -3x = 0$ or $x = 0$ so no solution here because it is not in line with $x < -3$.

(b)



3. (a) (i)



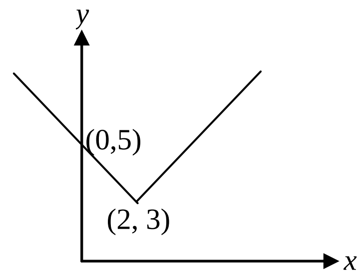
(ii) $y \geq -1$

(b) Solve the inequality $|2x - 4| < 8$ *)

For $x < 2$ *) becomes: $-2x + 4 < 8 \rightarrow -2x < 4$ or $x > -2$…….①

For $x \geq 2$ *) becomes $2x - 4 < 8 \rightarrow 2x < 12$ or $x < 6$………..②

Taking ① and ② together we get: $-2 < x < 6$.



4. (a) gf(4) =g(3 – 4) = g(–1) = $|2 \times(-1) -1| - |-5 \times(-1)| = 3 - 5 = -2$

(b) See graph on the right →

(c) $|3x - 6| = x + 1$*)

For $x > 2$ *) changes to $3x - 6 = x + 1$ or $2x = 7 \rightarrow \underline{x = 3.5}$

For $x \leq 2$ *) changes to $-3x + 6 = x + 1$ or $-4x = -5 \rightarrow \underline{x = 1.25}$

5. Solve: $|4 - 2x| \leq 20$ *)

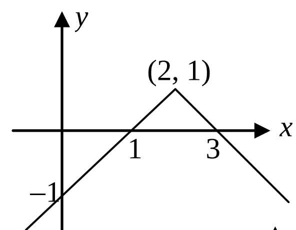
For $x < 2$ *) changes to $4 - 2x \leq 20 \rightarrow -2x \leq 16$ so $x \geq -8$. This results to $-8 \leq x < 2$… ①

For $x \geq 2$ *) changes to $2x - 4 \leq 20 \rightarrow 2x \leq 24$ so $x \leq 12$. This results to $2 \leq x \leq 12$ ………②

Taking ① and ② together we get $-8 \leq x \leq 12$.

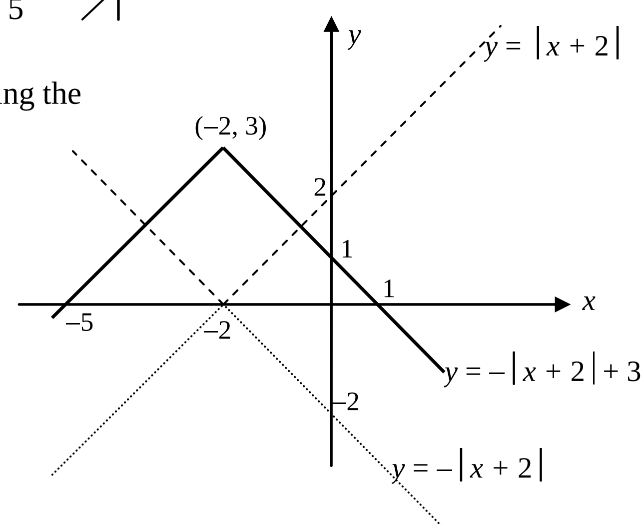
6. (a) $9^{|1-2x|} = 3^6$ can be rewritten as: $3^{2|1-2x|} = 3^6$ so $2|1 - 2x| = 6$ or $|1 - 2x| = 3$ this means $1 - 2x = 3$ or $\underline{x = -1}$ or $1 - 2x = -3$ this means $\underline{x = 2}$

(b) → → → →



(c) $|x - 2| < 3$ this means $-3 < x - 2 < 3$ or $-1 < x < 5$

7 (a) The dotted lines are not the solution but are assisting the drawing of the final graph. In the final exam it is better to erase them.



(b) $|3x - 2| > 4 \rightarrow 3x - 2 = 0$ for $x = \frac{2}{3}$

For $x \geq \frac{2}{3}$ then $|3x - 2| > 4$ changes into: $3x - 2 > 4$ or $3x > 6$ or $x > 2$

For $x < \frac{2}{3}$ the inequality $-(3x - 2) > 4$ or $-3x + 2 > 4$ so that $-3x > 2$ or $x < -\frac{2}{3}$

So $|3x - 2| > 4$ for $x > 2$ or $x < -\frac{2}{3}$.

8. (a) (i) Range of f is: $y \leq 4$ **(ii)** Range of g is $y \geqslant 0$

(b) For $x \leq -1$ or for $x \geqslant 1$

(c) The line passing through (–1, 4) and (1, 0) has gradient –2 so $a = 2$ then $b = -2$

(d) Both f^{-1} and g^{-1} do not exist. [Use the horizontal line test in the graph to see that there always originals with more than one image]

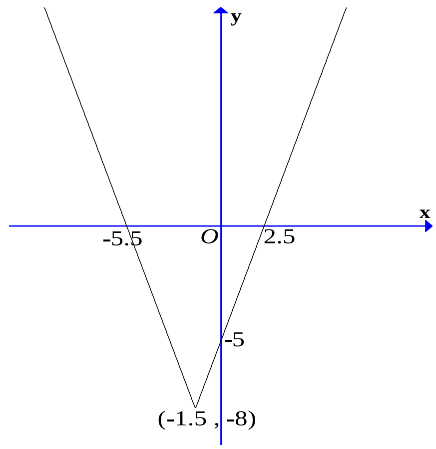
(e) The zeros of f are –3 and 1 so $f(x) = p(x + 3)(x - 1)$. (–1, 4) is on the graph so $4 = p(2) \times (-2)$ so $p = -1$

So $f(x) = -(x + 3)(x - 1)$ or $f(x) = -x^2 - 2x + 3$.

9.(a) $4|2x + 3| - 8 = 36 \rightarrow$ divide by 4: $|2x + 3| - 2 = 9$ or $|2x + 3| = 11$ so $2x + 3 = \pm 11$ so $\underline{x = 4 \text{ or } x = -7}$

(b) $|-3x + 2| < 1$. First solve: $-3x + 2 = 1$ that is for $x = \frac{1}{3}$ then solve $-(-3x + 2) = 1$ for $x = 1$ so $|-3x + 2| < 1$ for $\frac{1}{3} < x < 1$.

(c) See diagram.



Section 5 Particle mechanics

1. (a) $s = 10 - 10e^{-t} - \dfrac{1}{20}t$ so $v(t) = \dfrac{ds}{dt} = 10e^{-t} - \frac{1}{20}$ so $v(0) = 10 - 0.05 = \underline{9.95 \text{ m/s}}$

(b) $v(t) = 0 \rightarrow 10e^{-t} = 0.05$ so $e^{-t} = 0.005$ take natural log from both sides: $-t = \ln 0.005$ so $\underline{t = 5.298 \text{ s}}$
In three significant figures: $\underline{t = 5.30 \text{ s}}$

(c) $v(t) = 10e^{-t} - \frac{1}{20}$ so the acceleration is: $a(t) = \dfrac{dv}{dt} = -10e^{-t}$ so $a(5.30) = -10 \times 0.005 = \underline{-0.05 \text{ m/s}^2}$

2. (a) (i) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(3t - t^2) = 3 - 2t$

(ii) $s(t) = \int v(t)\,dt = \int (3t - t^2)\,dt = 1.5t^2 - \frac{1}{3}t^3 + c$ if $s(t)$ expresses the distance from flower F then $s(0) = 0$ so $c = 0$ the correct formula for the displacement is then $s(t) = 1.5t^2 - \frac{1}{3}t^3$.

(b) At B the speed will be 0, this is when $3t - t^2 = 0$ or $t(3 - t) = 0$ so $v = 0$ for $t = \underline{3\text{sec}}$

(c) $s(3) = 1.5 \times 9 - \frac{1}{3} \times 27 = 13.5 - 9 = \underline{4.5 \text{ m}}$

(d) $a(t) = 3 - 2t$ so $a(t) = 0$ for $t = 1.5$ sec. and $v(1.5) = 4.5 - 2.25 = \underline{2.25 \text{ m/s}}$

3. (a) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}35e^{-2t} = -70e^{-2t}$

(b) $s(t) = \int v(t)\,dt = \int 35e^{-2t}\,dt = -17.5e^{-2t} + c$ since $s(0) = 0$, $c = 17.5$ m so $s(t) = -17.5e^{-2t} + 17.5$.
$s(2) = 17.5 - 17.5e^{-4} = \underline{17.2 \text{ m}}$

4. (a) $v(t) = \int a(t)\,dt = \int (4t - 11)\,dt = 2t^2 - 11t + c$ since $v(0) = 14$ m/s, $c = 14$ so the velocity is $\underline{v(t) = 2t^2 - 11t + 14.}$

(b) $v(t) = 0$ so $2t^2 - 11t + 14 = 0$ or $2t^2 - 7t - 4t + 14 = 0$ or $t(2t - 7) - 2(2t - 7) = 0$ so that $(2t - 7)(t - 2) = 0$
So $v(t) = 0$ for $\underline{t = 2}$ seconds or $\underline{t = 3.5}$ seconds.

(c) $s(t) = \int v(t)\,dt = \int (2t^2 - 11t + 14)\,dt = \frac{2}{3}t^3 - 5.5t^2 + 14t + c$ since $s(0) = 50$; $c = 50$.
So $s(t) = \frac{2}{3}t^3 - 5.5t^2 + 14t + 50$

(d) $s(1) = \frac{2}{3} - 5.5 + 14 + 50 = 59\frac{1}{6}$ m.
So the distance travelled from O after 1 s is $59\frac{1}{6}$ m; the distance travelled in the first second is
$59\frac{1}{6} - 50 = 9\frac{1}{6}$ m [$s(t)$ is the distance to a fixed point O and at $t = 0$ the particle was already 50 m from O]

5. (a) (i) $v(t) = 12t - t^2 = t(12 - t)$ so for $t = 6$ there is a maximum velocity of $12 \times 6 - 36 = 36$m/s.

[Explanation: the graph of $v(t)$ is a parabola with zero's at $t = 0$ and $t = 12$. The line of symmetry is at $t = 6$ so $v(6)$ is the maximum velocity.]

(ii) $s(t) = \int v(t)\,dt = \int (12t - t^2)\,dt = 6t^2 - \frac{1}{3}t^3 + c$. With $c = 0$ because $s(0) = 0$ so $s(t) = 6t^2 - \frac{1}{3}t^3$

$OB = s(6) = 216 - 72 =$ <u>144 m.</u>

(b) (i) $s(t) = \int v(t)\,dt = \int (36 - 3t)\,dt = 36t - 1.5t^2 + c$ with $c = 0$. So $s(t) = 36t - 1.5t^2$.

Then $s(T) = 36T - 1.5T^2 = 120 \rightarrow$ divide by -1.5 you get: $T^2 - 24T + 80 = 0$.

(ii) $T^2 - 24T + 80 = 0 \rightarrow (T - 4)(T - 20) = 0$ so $T = 4$ s or $T = 20$s.

6. (a) $v = 20e^{-3t}$ then the acceleration is: $\dfrac{dv}{dt} = -60e^{-3t}$.

(b) $s(t) = \int v(t)\,dt = \int 20e^{-3t}\,dt = -\frac{20}{3}e^{-3t} + c$ with $s(0) = 0$ so that $c = \frac{20}{3} = 6\frac{2}{3}$. So $s(t) = -\frac{20}{3}e^{-3t} + 6\frac{2}{3}$

$s(1) = 6.335$ m

7. (a) Integrate: $\dfrac{dv}{dt} = -10$ then $v(t) = \int \dfrac{dv}{dt}\,dt = \int -10\,dt = -10t + c$

Since $v(0) = 12$ it means $c = 12$ so $v(t) = 12 - 10t$

(b) $s(t) = \int (12 - 10t)\,dt = 12t - 5t^2 + c$ with $c = 0$ because at moment $t = 0$ the stone is still at the ground. So $s(t) = 12t - 5t^2$.

(c) (i) $v = 0$

(ii) $v = 0$ so $12 - 10t = 0$ so $t = 1.2$ sec.

(iii) $s(1.2) = 12 \times 1.2 - 5 \times 1.2^2 = 7.2$ m.

<u>Section 6 Area and volume calculation through integration</u>

1. (a) Shaded area is $A = \int_0^4 y\,dx = \int_0^4 \sqrt{4-x}\,dx$. Use a substitution: $p = 4 - x \rightarrow \dfrac{dp}{dx} = -1$ so $dp = -dx$

When $x = 0$ then $p = 4$ and when $x = 4$ then $p = 0$ so $A = -\int_4^0 p^{0.5}\,dp = -\left[\dfrac{1}{1.5}p^{1.5}\right]_4^0 = -(0 - \dfrac{8}{1.5}) = 5\frac{1}{3}$

(b) $V = \pi\int_0^4 y^2\,dx = \pi\int_0^4 (4 - x)\,dx = \pi\left[4x - \frac{1}{2}x^2\right]_0^4 =$ <u>8π</u>

2. (a) First make y^2 the subject of the formula: $y^2 = x - 1$, required volume is: $V = \pi\int_1^2 y^2\,dx = \pi\int_1^2 (x-1)\,dx =$

$= \pi\left[\frac{1}{2}x^2 - x\right]_1^2 = \pi\{(\frac{1}{2} \times 4 - 2) - (\frac{1}{2} \times 1 - 1)\} =$ <u>$\frac{1}{2}\pi$.</u>

(b) $y^2 = x - 1 \rightarrow y = \sqrt{x-1} \rightarrow \dfrac{dy}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$ or $\dfrac{dy}{dx} = \dfrac{1}{2(x-1)^{0.5}}$ so the gradient at B is $\dfrac{1}{2(2-1)^{0.5}} = \frac{1}{2}$

The equation is $y = 0.5x + c$. $B(2, 1)$ on the line so $1 = 0.5 \times 2 + c$ so $c = 0$. So the equation is <u>$y = 0.5x$</u>

3. (a) Point A : y coordinate is zero so $9 - x = 0$ so $x = 9 \rightarrow A(9, 0)$

Point B : x coordinate is zero so $y^2 = 9$ so $y = 3 \rightarrow B(0, 3)$.

(b) Area shaded region = $\int_0^9 y\,dx = \int_0^9 \sqrt{9-x}\,dx$ use substitution 9 – *x* = *p* so that $\frac{dp}{dx} = -1$ or *dx*= –*dp*

Area shaded region = $\int_9^0 -p^{\frac{1}{2}}\,dp = \left[-\tfrac{2}{3}p^{1.5}\right]_9^0 = -\tfrac{2}{3}\left[p^{1.5}\right]_9^0 = -\tfrac{2}{3}\{0 - 9^{1.5}\} = -\tfrac{2}{3}\times(-27) =$ <u>18 sq units.</u>

(c) Generated volume = $\int_0^3 \pi x^2 dy = \pi\int_0^3 (9-y^2)^2 dy = \pi\int_0^3 (81 - 18y^2 + y^4)\,dy = \pi\left[81y - 6y^3 + 0.2y^5\right]_0^3 =$

$\pi\{243 - 162 + 0.2\times 243\} = 129.6\pi = 130\pi$

4. (a) $\frac{dy}{dx} = \frac{d}{dx}\sqrt{3x+4}\;dx = \frac{1.5}{\sqrt{3x+4}}$ the value of $\frac{dy}{dx}$ at *x* = 4 is 1.5 ÷ √16 = $\frac{3}{8}$. This means the tangent has equation

$y = \frac{3}{8}x + c$. The tangent passes through *T*(4,4) so $4 = \frac{3}{8}\times 4 + c$ this means *c* = 2½ so

the equation is $y = \frac{3}{8}x + 2.5$.

(b) Area shaded region is: Area trapezium – $\int_0^4 \sqrt{3x+4}\,dx$ to integrate a substitution is used: *p* = 3*x* + 4;

$\frac{dp}{dx} = \frac{d}{dx}(3x+4) = 3$ so $dp = 3dx$

Area shaded region is = ½ *h*(*a* + *b*) – $\frac{1}{3}\int_4^{16} p^{\frac{1}{2}}dp = \frac{1}{2}\times 4(2.5 + 4) - \frac{1}{3}\left[\frac{2}{3}p^{1.5}\right]_4^{16} = 13 - \frac{2}{9}\{64-8\} = 13 - 12\frac{4}{9} = \frac{5}{9}$

(c) $V = \pi\int_0^4 y^2\;dx = \pi\int_0^4 (3x+4)\,dx = \pi\left[1.5x^2 + 4x\right]_0^4 = \pi\{24+16-0\} =$ <u>40π</u>

5. (a) For *A*, *y* = 0 so <u>*A*(–16, 0)</u> and for *B*, *x* = 0 so *y* = +√16 = 4. This means <u>*B*(0, 4)</u>

(b) Area shaded region = $\int_{-16}^0 y\,dx = \int_{-16}^0 \sqrt{x+16}\;dx$ for this integration a substitution can be used: *p* = *x* + 16.

$dp = dx$, but the boundary points change: Area shaded region = $\int_0^{16} p^{\frac{1}{2}}\;dp = \left[\frac{2}{3}p^{1.5}\right]_0^{16} = \frac{2}{3}\{16\times 4 - 0\} = 42\frac{2}{3}$

(c) Generated volume = $\pi\int_0^4 x^2\;dy = \pi\int_0^4 (y^2-16)^2 dy = \pi\int_0^4 (y^4 - 32y^2 + 256)dy = \pi\left[0.2y^5 - \frac{32}{3}y^3 + 256y\right]_0^4 =$

$= \pi[0.2\times 1024 - 2048 \div 3 + 1024] = 1024\pi(0.2 - \frac{2}{3} + 1) = 1024\pi\times\frac{8}{15} = 546\frac{2}{15}\pi$

6. (a) $y = x + \frac{3}{x}$ ..*differentiate with respect to *x*: $\frac{dy}{dx} = 1 - \frac{3}{x^2}$ so the tangent at *x* =1 to the curve will have gradient

1 – 3 ÷ 1 = –2. [This is possible if you check the graph in the diagram.] The tangent passes through (1, 4) [substitute *x* = 1 in *] So the equation of the tangent is *y* = –2*x* + *c* or 4 = –2 + *c* → *c* = 6.
So the equation of the tangent is <u>*y* = –2*x* + 6.</u>
[Again the diagram gives further assurance that this is the correct answer.]

(b) Area shaded region = area under the curve – area trapezium = $\int_3^4 (x + \frac{3}{x})\,dx - \frac{1}{2}\times 1\times(3+4) =$

$[\frac{1}{2}x^2 + 3\ln x]_3^4 - 3.5 = 8 + 3\ln 4 - (4.5 + 3\ln 3) - 3.5 = 3\ln 4 - 3\ln 3 =$ <u>0.863</u>

(c) Requested volume = $\pi\int_3^4\left(x+\frac{3}{x}\right)^2 dx = \pi\int_3^4 (x^2 + 6 + \frac{9}{x^2})dx = \pi\left[\frac{1}{3}x^3 + 6x - \frac{9}{x}\right]_3^4 =$

$\pi\{\frac{1}{3}\times 64 + 24 - 2.25 - (9 + 18 - 3)\} = \pi(21\frac{4}{12} - 2\frac{3}{12} = 19\frac{1}{12}\,\pi$

7. (a) For A ,$y = 0$ so $x^2 = 16 - y$ so $x = 4$ this means $A(4, 0)$. For B, $x = 0$ so $y = 16$ this means $B(0, 16)$.

(b) Make y subject: $y = 16 - x^2$; the shaded area = $\int_0^4 (16 - x^2)dx = [16x - \frac{1}{3}x^3]_0^4 = 64 - \frac{64}{3} = 42\frac{2}{3}$

(c) Volume = $\pi\int_0^{16} x^2 dy = \pi\int_0^{16}(16 - y)\ dy = \pi[16y - \frac{1}{2}y^2]_0^{16} = \pi\times(256 - 128) = 128\pi$

8. (a) $y = \sqrt{2x+1}$; then $\frac{dy}{dx} = \frac{2\times\frac{1}{2}}{\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$ substitute $x = 4$ and you get for the gradient $\frac{1}{3}$

(b) Calculate: $\pi\int_0^4\left(\sqrt{2x+1}\right)^2 dx = \pi\int_0^4 (2x+1)dx = \pi\left[x^2 + x\right]_0^4 = 20\pi$

(c) The integration $\int_0^4\left(\sqrt{2x+1}\right)dx$ can best be done through a substitution: let $p = 2x + 1$then $\frac{dp}{dx} = 2$ so ½$d\,p = dx$

and the boundaries change from 1 to 9.

Requested area = $\int_0^4\left(\sqrt{2x+1}\right)dx$ – area trapezium = $\int_1^9\left(\sqrt{p}\right)\frac{1}{2}dp = \frac{1}{2}\left[\frac{2}{3}p^{1.5}\right]_1^9 = \frac{1}{3}\{27 - 1\} = 8\frac{2}{3}$ – ½.4(1 + 3) = $\frac{2}{3}$

9. (a) $A(0,4)$ and $B(1.3)$

(b) $\frac{dy}{dx} = -2x$ so the tangent has gradient −4. So the eq. is $y = -4x + c$. (2,0) on the curve so $c = 8$.

The tangent has equation $\underline{y = -4x + 8}$.

(c) Area shaded region = $\int_0^1 (4 - x^2)dx - 3 = \left[4x - \frac{1}{3}x^3\right]_0^1 - 3 = 4 - \frac{1}{3} - 3 = \frac{2}{3}$

(d) Volume = $\pi\int_3^4 x^2 dy = \pi\int_3^4 (4 - y)dy = \pi\left[4y - \frac{1}{2}y^2\right]_3^4 = \pi\{(16 - 8) - (12 - 4.5)\} = \pi(8 - 7.5) = 0.5\pi$

Section 7 3D and 2D vectors

1. (a) Given: $\overrightarrow{OA} = 3\mathbf{i} + 2\mathbf{k}$; $\overrightarrow{OB} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{OC} = -2\mathbf{i} + 10\mathbf{j} - 3\mathbf{k}$ then

$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -(2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + 3\mathbf{i} + 2\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(b) Use the scalar product: $\overrightarrow{BA}\bullet\overrightarrow{BC} = \left|\overrightarrow{BA}\right|\times\left|\overrightarrow{BC}\right|\times\cos A\hat{B}C$ = product components

$\overrightarrow{BA} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -(2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + (-2\mathbf{i} + 10\mathbf{j} - 3\mathbf{k}) = -4\mathbf{i} + 12\mathbf{j} - 8\mathbf{k}$

Cos $ABC = \frac{(-4)\times 1 + 2\times 12 + (-3)\times(-8)}{\sqrt{1^2 + 2^2 + (-3)^2}\times\sqrt{(-4)^2 + 12^2 + (-8)^2}} = \frac{44}{\sqrt{14}\times\sqrt{224}} = \frac{44}{56} = \frac{11}{14}$ so $\angle ABC = 38.2^{\circ}$

(c) $\overrightarrow{OD} = \frac{1}{5}\overrightarrow{AC}$ so $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -(3\mathbf{i} + 2\mathbf{k}) - 2\mathbf{i} + 10\mathbf{j} - 3\mathbf{k} = -5\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}$ so

$\overrightarrow{OD} = \frac{1}{5}\overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ so D(−1, 2, −1)

2. (a) If A is (2, –2, 2) then $\overrightarrow{OA} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ as well $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} - 3\mathbf{i} + 2\mathbf{j} - \mathbf{k} = -\mathbf{i} + \mathbf{k}$
So <u>$B(-1, 0, 1)$</u>

(b) (i) AB and CD are parallel → $q\,\overrightarrow{AB} = \overrightarrow{CD}$ so $q(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 9\mathbf{i} + p\mathbf{j} + 3\mathbf{k}$ looking at the **i** component one can conclude $q = -3$ this means <u>$p = -6$.</u>

(ii) AB and CD are perpendicular → the scalar product is zero so $-3 \times 9 + 2 \times p + (-1) \times 3 = 0$ or $-27 + 2p - 3 = 0$ or <u>$p = 15$.</u>

(iii) $\overrightarrow{AB} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{CD} = 9\mathbf{i} + 11\mathbf{j} + 3\mathbf{k}$.

Use the formula: $\cos\alpha = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|a||b|} = \dfrac{-27 + 22 - 3}{\sqrt{(-3)^2 + 2^2 + (-1)^2} \times \sqrt{9^2 + 11^2 + 3^2}} = \dfrac{-8}{\sqrt{14}\sqrt{211}}$

The required angle is $\cos^{-1}\left(\dfrac{-8}{\sqrt{14}\sqrt{211}}\right) = $ <u>98.5°</u>

3. (a) A has coordinates (–3, 3, 3) → $\mathbf{OA} = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ then $\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} - 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} = -5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ so B has coordinates <u>(–5, 6, –3).</u>

(b) (i) If AB and CD are parallel then $p\mathbf{AB} = \mathbf{CD}$ or $p(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 8\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$. Comparing the **i** component you get $p = -4$ so <u>$t = 24$</u>.

(ii) Perpendicular → scalar product is zero so $\mathbf{AB} \bullet \mathbf{CD} = (-2) \times 8 + 3 \times (-12) + (-6) \times t = 0$ or $-16 - 36 - 6t = 0$ or $6t = -52 \rightarrow t = -8\tfrac{2}{3}$

(c) Use the formula: $\cos\alpha = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|a||b|} = \dfrac{-16 - 36 + 6}{\sqrt{4 + 9 + 36} \times \sqrt{64 + 144 + 1}} = \dfrac{-46}{\sqrt{49}\sqrt{209}} = -0.45455522$

The requested angle is $\cos^{-1}(-0.45455522) = 117.0^\circ$

4. (a) $\mathbf{OA} \bullet \mathbf{OB} = |\mathbf{OA}|\,|\mathbf{OB}| \cos AOB = a_1b_1 + a_2b_2 \rightarrow \cos AOB = \dfrac{a_1b_1 + a_2b_2}{|\overrightarrow{OA}||\overrightarrow{OB}|}$ angle $AOB = \cos^{-1}\left(\dfrac{a_1b_1 + a_2b_2}{|\overrightarrow{OA}||\overrightarrow{OB}|}\right) =$

$\text{Cos}^{-1}\left(\dfrac{10 \times 1 + 5 \times 3}{\sqrt{125} \times \sqrt{10}}\right) = $ <u>45°</u>

(b) AB = AO + OB = $\begin{pmatrix} -10 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 \\ -2 \end{pmatrix}$ so $|\mathbf{AB}| = \sqrt{(81 + 4)} = \sqrt{85} = 9.22$

(c) AC = AO + OC = $\begin{pmatrix} -10 \\ -5 \end{pmatrix} + \begin{pmatrix} p \\ 3p \end{pmatrix} = \begin{pmatrix} p - 10 \\ 3p - 5 \end{pmatrix}$.

$|\overrightarrow{OC}| = |\overrightarrow{AC}| \rightarrow \sqrt{(p^2 + 9p^2)} = \sqrt{[(p - 10)^2 + (3p - 5)^2]}$ after squaring both sides you get:

$10p^2 = p^2 - 20p + 100 + 9p^2 - 30p + 25$ or $50p = 125$ so that <u>$p = 2\tfrac{1}{2}$</u>

5. (a) $\cos\alpha = \dfrac{a_1c_1 + a_2c_2 + a_3c_3}{|a||c|} = \dfrac{4 - 12 + 3}{\sqrt{4 + 4 + 1} \times \sqrt{4 + 36 + 9}} = \dfrac{-5}{\sqrt{9}\sqrt{49}} = \dfrac{-5}{21}$ requested angle is $\cos^{-1}\left(\frac{-5}{21}\right) = $ <u>103.8°</u>

(b) AB = AO + OB = $\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} p \\ 3 \\ p + 1 \end{pmatrix} = \begin{pmatrix} p - 2 \\ 5 \\ p \end{pmatrix}$ and **AC = AO + OC =** $\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{AC}$ = $\underline{40 + 2p}$

(c) $BAC = 90° \rightarrow \overrightarrow{AB} \cdot \overrightarrow{AC} = 0$ or $40 + 2p = 0$ so that $\underline{p = -20.}$

6. (a) $\underline{C(2, -1, 0)}$

(b) BA = BO + OA = –(2j + 3k) + (–i – 2j + k) = $\underline{-\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}.}$

BC = BO + OC = –(2j + 3k) + (2i – j) = $\underline{2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}}$

(c) Angle $ABC = \cos^{-1}\left(\dfrac{2\times(-1)+(-3)\times(-4)+(-2)\times(-3)}{\sqrt{1+16+4}\times\sqrt{4+9+9}}\right) = \cos^{-1}\left(\dfrac{-2+12+6}{\sqrt{21}\times\sqrt{22}}\right) = \cos^{-1}\left(\dfrac{16}{\sqrt{462}}\right)$ = 0.731 rad

[1.3 rad. is correct as well since angles are to rounded off to one decimal place]

7. (a) use the scalar product: $\cos A\hat{O}B = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|a||b|}$ = (2 – 4) ÷ (√21 × √5) so angle AOB = 101°

(b) First find vector ***AB*** and ***AC.***

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \begin{pmatrix}-2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\5\end{pmatrix} \text{ and } \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \begin{pmatrix}0\\c\\-1\end{pmatrix} = \begin{pmatrix}1\\-2+c\\3\end{pmatrix}$$

$BAC = 90°$ so the scalar product is zero: $-1 \times 1 + (-2)\times(-2 + c) + 3 \times 5 = 0$ work out brackets:
$-1 + 4 - 2c + 15 = 0 \rightarrow -2c = -18$ so that $c = 9$

(c) $\overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD} = \begin{pmatrix}-2\\0\\-1\end{pmatrix} + \begin{pmatrix}d\\-2\\1\end{pmatrix} = \begin{pmatrix}d-2\\-2\\0\end{pmatrix}$ then $\sqrt{(d-2)^2+4} = 2$ so that $\underline{d = 2}$

8. (a) $|\overrightarrow{OA}| = \sqrt{[4^2 + (-2)^2 + (2\sqrt{11})^2]} = \sqrt{[16 + 4 + 44]} = \sqrt{64} = 8.$

(b) Then $\overrightarrow{OB}$ = 2[4**i** – 2**j** + 2√11**k**] = 8**i** – 4**j** + 4√11**k** so B(8, –4, 4√11)

(c) (i) $\overrightarrow{OA}.\overrightarrow{OC} = 0$ so $a_1b_1 + a_2b_2 + a_3b_3 = |\overrightarrow{OA}| \times |\overrightarrow{OB}| \times \cos\angle AOC = 0$

$a_1b_1 + a_2b_2 + a_3b_3 = 0$ or $4 \times 2 + (-2) \times 2p = 0$ or $8 = 4p \rightarrow \underline{p = 2}$

(ii) cos 60° = 0.5 so $4 \times 2 + (-2) \times 2p = 8 \times \sqrt{(4 + 4p^2)} \times 0.5$ rewrite: $8 - 4p = 4\sqrt{(4 + 4p^2)}$

Divide by 4: $2 - p = \sqrt{(4 + 4p^2)}$ square both sides: $(2 - p)^2 = 4 + 4p^2$

Work out the brackets and write to standard form: $4 - 4p + p^2 = 4 + 4p^2 \rightarrow 3p^2 + 4p = 0.$

So $p = 0$ or $p = -1\frac{1}{3}$

9. (a) Use the scalar product: $\overrightarrow{OA} \cdot \overrightarrow{OB} = |\overrightarrow{OA}| \times |\overrightarrow{OB}| \times \cos AOB = a_1b_1 + a_2b_2 + a_3b_3$ so $\cos AOB = \dfrac{-2-2}{\sqrt{21}\times\sqrt{5}}$ this

means angle $AOB = \cos^{-1}(-4/\sqrt{105}) = 113°$.

(b) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}3\\-3\\4\end{pmatrix}$ and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \begin{pmatrix}0\\c\\1\end{pmatrix} = \begin{pmatrix}1\\c-2\\5\end{pmatrix}$

The scalar product is then zero, so $3 - 3(c - 2) + 20 = 0$ or $3(c - 2) = 23$ ➔ $c - 2 = 7\frac{2}{3}$ so that $c = 9\frac{2}{3}$.

10 (a) (i) $OX = 5\mathbf{j} + 3\mathbf{k}$ **(ii)** $XB = 6\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$ **(iii)** $AC = 6\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}$

(b) Then $XO = -5\mathbf{j} - 3\mathbf{k}$ so $\cos O\hat{X}B = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|a||b|} = \dfrac{25+9}{\sqrt{34}\times\sqrt{70}}$ so angle OXB = 0.80 radians.

Section 8 Factor and remainder theorem

1. (a) Let $f(x) = x^3 + 2x^2 - 7x - 12$ check $f(-3) = -27 + 18 + 21 - 12 = 0$ indeed so $(x + 3)$ is a factor.

$$
\begin{array}{rl}
 & x^2 - x - 4 \\
x + 3 \,\big) & x^3 + 2x^2 - 7x - 12 \\
- & \underline{x^3 + 3x^2} \\
 & -x^2 - 7x \\
- & \underline{-x^2 - 3x} \\
 & -4x - 12 \\
- & \underline{-4x - 12} \\
 & 0
\end{array}
$$

Based on the long division we now know:
$f(x) = x^3 + 2x^2 - 7x - 12 = (x + 3)(x^2 - x - 4)$
The quadratic part has to be solved: $x^2 - x - 4 = 0$
or $x = \dfrac{1 \pm \sqrt{1+16}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}$ so the solutions are:
$\underline{-3;\ 2.56 \text{ or } -1.56}$

(b) $f(x) = x^3 + ax^2 + bx - 3$ has a factor $(x + 1) \rightarrow f(-1) = 0$ so $-1 + a - b - 3 = 0$ rewritten: $a - b = 4$….①
When $f(x)$ is divided by $(x + 2)$, it has the same remainder as when it is divided by $(x - 2)$. $\rightarrow f(-2) = f(2)$.
This will provide another relation between a and b: $-8 + 4a - 2b - 3 = 8 + 4a + 2b - 3$ or $-16 = 4b$
So that $\underline{b = -4}$ and using ① we get $\underline{a = 0.}$

2. (a) Work out the long division:

$$
\begin{array}{rl}
 & 6x^2 - 19x + 9 \\
x + 1 \,\big) & 6x^3 - 13x^2 - 10x + 24 \\
- & \underline{6x^3 + 6x^2} \\
 & -19x^2 - 10x \\
- & \underline{-19x^2 - 19x} \\
 & 9x + 24 \\
- & \underline{9x + 9} \\
 & 15
\end{array}
$$

Conclusion:
$\underline{a = 6;\ b = -19;\ c = 9}$
and $\underline{R = 15}$

(b) $f(2) = 6 \times 8 - 13 \times 4 - 10 \times 2 + 24 = 72 - 52 - 20 = 0$ so $(x - 2)$ is a factor of $f(x)$.

(c) Work out the long division:

$$
\begin{array}{rl}
 & 6x^2 - x - 12 \\
x - 2 \,\big) & 6x^3 - 13x^2 - 10x + 24 \\
- & \underline{6x^3 - 12x^2} \\
 & -x^2 - 10x \\
- & \underline{-x^2 + 2x} \\
 & -12x + 24 \\
- & \underline{-12x + 24} \\
 & 0
\end{array}
$$

Conclusion:
$6x^3 - 13x^2 - 10x + 24 = (x - 2)(6x^2 - x - 12) =$
$(x - 2)(6x^2 - 9x + 8x - 12) =$
$(x - 2)[3x(2x - 3) + 4(2x - 3)] =$
$\underline{(x - 2)(2x - 3)(3x + 4)}$

3. (a) $f(-1) = -2 + 3 + 5 - 6 = 0$ so $(x + 1)$ is a factor of $f(x)$.

(b) Work out:

$$
\begin{array}{rl}
 & 2x^2 + x - 6 \\
x + 1 \,\big) & 2x^3 + 3x^2 - 5x - 6 \\
- & \underline{2x^3 + 2x^2} \\
 & x^2 - 5x \\
- & \underline{x^2 + x} \\
 & -6x - 6 \\
- & \underline{-6x - 6} \\
 & 0
\end{array}
$$

This means $2x^3 + 3x^2 - 5x - 6 = (x + 1)(2x^2 + x - 6)$ so that $\underline{a = 2}$; $\underline{b = 1}$ and $\underline{c = -6}$

(c) $(x + 1)(2x^2 + 4x - 3x - 6) = 0$
$(x + 1)[2x(x + 2) - 3(x + 2)] = 0$
$(x + 1)(x + 2)(2x - 3) = 0$ so $\underline{x = -1 \text{ or } x = -2 \text{ or } x = 1\tfrac{1}{2}.}$

4. **(a)** substitute $x = -2$ in $x^2 - px + 1$, you will get: $4 + 2p + 1 = 5 + 2p = 3$ so $\underline{p = -1}$

(b) **(i)** $x^2 - 3x + 1 = 11 \rightarrow x^2 - 3x - 10 = 0$ or $(x - 5)(x + 2) = 0$ so that $\underline{x = 5 \text{ or } x = -2.}$

(ii) using **(i)** we can write down: $\sqrt{a-1} = 5$ or $\underline{a = 26}$.

[The root -2 will not give a solution since a square root always gives a positive answer.]

5. **(a)** So $f(2) = -6$ or $8 + 8 + 2a - 8 = -6 \rightarrow 2a = -14$ or $\underline{a = -7}$ So $f(x) = x^3 + 2x^2 - 7x - 8$

(b) **(i)** Work out $f(-1)$: $-1 + 2 + 7 - 8 = 0$ indeed so $(x + 1)$ is factor.

(ii) [One can work this out with a process of long division, but that takes time. Smarter is the following:] $1 \times c = -8$ so $\underline{c = -8}$. For b you can say: $b \times 1 + c \times 1 = -7$ [Because $(x + 1)(x^2 + bx + c) = x^3 + 2x^2 - 7x - 8$]. So $b - 8 = -7$, this means $\underline{b = 1}$. So $f(x) = (x + 1)(x^2 + x - 8)$

(iii) $\underline{x = -1}$ is one solution; $x^2 + x - 8 = 0$ is the other part of the equation . Solve with *abc*-formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4.1.(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{33} = \underline{2.37} \text{ or } \underline{x = -3.37}$$

(c) **(i)** $f'(x) = 3x^2 + 4x - 7$ turning points have x–coordinates $3x^2 + 4x - 7 = 0$ or $3x^2 + 7x - 3x - 7 = 0$ so that $x(3x + 7) - 1(3x + 7) = 0 \rightarrow (3x + 7)(x - 1) = 0 \rightarrow x = 1$ or $x = -{}^7/_3$.

(ii) $f''(x) = 6x + 4$ but $f''(-{}^7/_3) < 0$ for $x = -{}^7/3$ there is a maximum and $f''(1) > 0$ so for $x = 1$ there is a minimum.

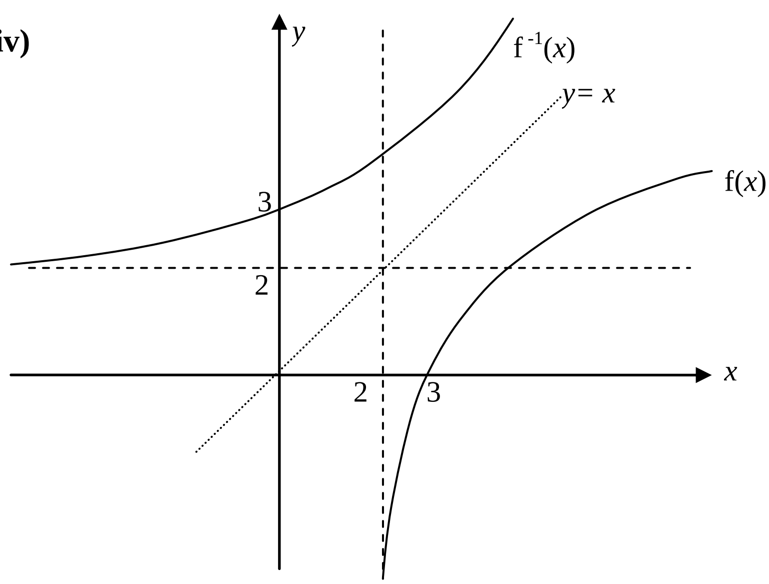
6. **(a)** **(i)** $x - 2 > 0$ so $a = 2$.

(ii) range is $\mathbb{R}$

(iii) $f(x) = \ln(x - 2)$ or $y = \ln(x - 2)$ this means $e^y = x - 2$ or $e^y + 2 = x$ swop x and y: $y = e^x + 2$

So $f^{-1}(x) = e^x + 2$

(iv)



(b) **(i)** $gf(4) = g(\ln 2) = \ln 2 + 3 = \underline{3.69}$

(ii) $fg(-3) = f(0)$ but f is only defined for $x > 2$ so $fg(-3)$ does not exist.

7. **(a)** When $f(x)$ is divided by $(x + 2)$, the remainder is $5 \rightarrow f(-2) = 5$ or

$5 = -(-2)^3 + (-2)^2 + 5(-2) + c \rightarrow 5 = 8 + 4 - 10 + c$ so that $c = 3$

(b) **(i)** $(x + 1)$ is a factor then $f(-1) = 0$ substitute in $f(x) = -x^3 + x^2 + 5x + 3$ it becomes:

$0 = 1 + 1 - 5 + 3$ which is true. Conclusion: $(x + 1)$ of $f(x)$.

(ii) Work out a long division:

$$\begin{array}{r} -x^2 + 2x + 3 \\ x+1 \overline{)\, -x^3 + x^2 + 5x + 3} \\ -\quad \underline{-x^3 - x^2} \\ 2x^2 + 5x \end{array}$$

8. **(a)** Work out f(–1) = –10 – 1 + 17 – 6 = 0 so (x + 1) is a factor.

(b) Divide:

$$\begin{array}{r} 10x^2 - 11x - 6 \\ x+1 \overline{)\, 10x^3 - x^2 - 17x - 6} \\ -\quad \underline{10x^3 + 10x^2} \\ -11x^2 - 17x \\ -\quad \underline{-11x^2 - 11x} \\ -6x - 6 \\ -\quad \underline{-6x - 6} \\ 0 \end{array}$$

So $a = 10$; $b = -11$ and $c = -6$

Alternatively:
The value of a, b and c can as well be obtained by inspection:
$f(x) = (x + 1)(ax^2 + bx + c) = 10x^3 - x^2 - 17x - 6$
$a = 10$ can be seen immediately since $x \times ax^2$ is the only product giving the $10x^3$. As well $1 \times c = 6$ delivers $c = 6$. The b is a bit more difficult since there are two products contributing to $-17x$.

(c) So $f(x) = 10x^3 - x^2 - 17x - 6 = (x + 1)(10x^2 - 11x - 6) = 10x^2 - 15x + 4x - 6) =$
$(x + 1)[5x(2x - 3) + 2(2x - 3)] = (x + 1)(2x - 3)(5x + 2)$
$f(x) = 0$ for $x = -1$ or $x = 1.5$ or $x = -0.4$

$$\begin{array}{r} -\quad \underline{2x^2 + 2x} \\ 3x + 3 \\ -\quad \underline{3x + 3} \\ 0 \end{array}$$

This means $-x^3 + x^2 + 5x + 3 = (x + 1)(-x^2 + 2x + 3) = (x + 1)(-x + 3)(x + 1) = (x + 1)^2(-x + 3)$

(c) **(i)** Work out $f'(x) = 0$; $f'(x) = -3x^2 + 2x + 5 = 0$ or $(-3x + 5)(x + 1) = 0$ this means there is a turning point for $x = -1$ and for $x = 1\frac{2}{3}$

(ii) Work out $f''(x)$; $f''(x) = -6x + 2$, $f''(-1) > 0$ so $x = -1$ is a minimum; $f''(1\frac{2}{3}) < 0$ so $x = 1\frac{2}{3}$ is maximum

9. It is given that $f(x) = x^2 - px - 1$, where p is a constant.

(a) Then f(3)= 0 or f(3) = $9 - 3p - 1 = -12$or $3p = 20$ so that $p = \frac{20}{3} = 6\frac{2}{3}$

(b) **(i)** $p = 1$, so $x^2 - x - 1 = 5$ or $x^2 - x - 6 = 0$ or $(x - 3)(x + 2) = 0 \rightarrow x = 3$ …①or $x = -2$….②.

(ii) $\left(\sqrt{a-2}\right)^2 - \sqrt{a-2} - 1 = 5 \rightarrow$ only ① will give a solution since a square root has a positive outcome.
$\sqrt{(a - 2)} = 3$ or $a - 2 = 9$ so $a = 11$.

10. **(a)** $f(x) = (ax^2 + bx + c)(x + 1) + R = ax^3 + (a + b)x^2 + (b + c)x + c + R$ this is identical to
$f(x) = 6x^3 - 7x^2 - 28x + 20$ so $a = 6$; $b = -13$; $c = -15$ and $R = 35$.

(b) Find f(–2): f(–2) = – 48 – 28 + 56 + 20 = 0 so $x + 2$ is a factor.

(c) One can write down immediately: $f(x) = (x + 2)(6x^2 + px + 10) = 6x^3 - 7x^2 - 28x + 20$ with $p = -19$
So $f(x) = (x + 2)(6x^2 - 19x + 10) = (x + 2)(6x^2 - 15x - 4x + 10) = (x + 2)[3x(2x - 5) - 2(2x - 5) =$
$(x + 2)(2x - 5)(3x - 2)$ so $f(x) = (x + 2)(2x - 5)(3x - 2)$

11. (a) So f(2) = −12 or 8 – 14 + c = −12 so that c = −6 indeed.

(b) $f(x) = x^3 - 7x - 6$. One can easily see that f(−1) = 0 so x + 1 is a factor.

Then $x^3 - 7x - 6 = (x + 1)(ax^2 + bx + d)$ …*

One can easily see that a = 1 and d = −6 so * changes into: $x^3 - 7x - 6 =$

The −7 is generated by multiplying $bx \times 1$ and $x \times (-6)$ so b – 6 = −7 so b = −1.

$f(x) = (x + 1)(x^2 - x - 6) = \underline{(x + 1)(x - 3)(x + 2)}.$ [The way shown here is a shortcut, one can use long division as well.]

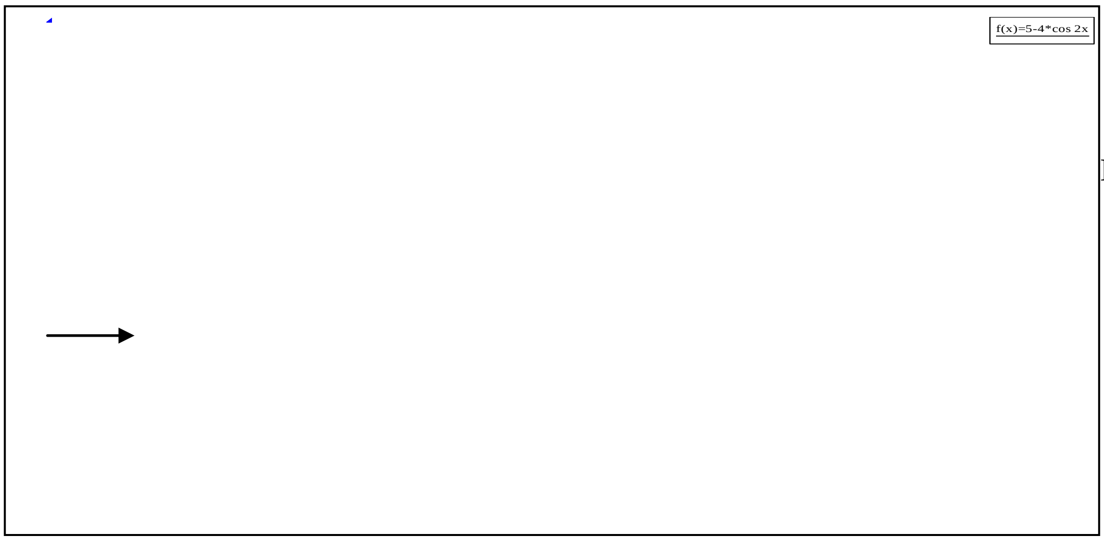
(c) (i) $f(x) = x^3 - 7x - 6 \rightarrow f'(x) = 3x^2 - 7$. Solve $f'(x) = 0$ or $3x^2 - 7 = 0$ or $3x^2 = 7$ or $x = \pm\sqrt{2\frac{1}{3}}$

(ii) $f''(x) = 6x$ this means for $x = -\sqrt{2\frac{1}{3}}$ there is a maximum value and for $x = \sqrt{2\frac{1}{3}}$ there is a minimum value.

Section 9 Trigonometric functions

1. (a) Amplitude is <u>4</u> [Half the range]; period is <u>180º</u>. [2x varies from 0º to 360º when x varies from 0º to 180º.]

(b)



Maximum points: (90º, 9)
And (270º , 9)

(c) 5 – 4cos 2x = 3 or – 4cos 2x = –2 so that cos 2x = ½ this means 2x = 60º or <u>x = 30º.</u>

2. (a) T(½π, –2)

(b) period of π radians.

(c) $-2 \leq y \leq 2$

(d) $D\left(1\frac{1}{3}\pi, \sqrt{3}\right)$

3. (a) (i) a = 2 and b = 1 **(ii)** range is $-2 \leq y \leq 2$

(b) (i) p =1 and q = 3 **(ii)** period is 120º

(c) f(x)g(x) < 0 for –30º < x < 0º or for 30º < x < 90º or for 150º < x < 180º.

4. (a) a = –2; b = 2; c = 2 **(b)** g : $x \mapsto -\tan 2x$

5. (a) (i) a = 2; b = 3; c = 1

(ii) 2sin (3x) + 1 = 0 → sin 3x = – ½ so 3x = –30º + k × 360º with $k \in \mathbb{Z}$ this means x = –10º + 120k…①

or 3x = – 150º + k × 360º with $k \in \mathbb{Z}$ this means x = –50º + 120k…..②

① gives that <u>x = 110º</u> when you take k = 1 but ② gives x = 70º. After checking the graph → <u>x = 70º</u>

(b) (i) $p = q \rightarrow 4 \sin x - 3 \cos x = 4 \cos x + 3\sin x$ or $\sin x - 7\cos x = 0$ divide by $\cos x$: $\tan x = 7$
so that $x = 1.43$ rad.

(ii) $p^2 + q^2 = (4 \sin x - 3 \cos x)^2 + (4 \cos x + 3\sin x)^2 =$
$16\sin^2 x - 24\sin x \cos x + 9\cos^2 x + 16\cos^2 x + 24\sin x \cos x + 9\sin^2 x = 16 + 9 = 25.$

6. (a) $a = 3$; $b = 2$; $c = -2$ and $d = 1$

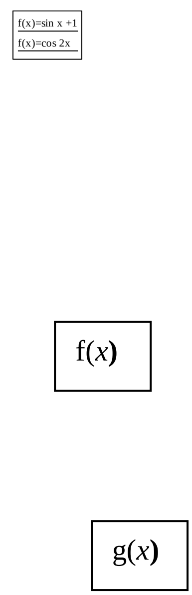
(b) $A(180^o, 2)$

(c) period is 180º (graph repeats itself after 180º.)

(d) Range of $y = c \cos dx$ is $-2 \leq y \leq 2$.

(e) Two possible answers: ① $y = 3 \cos 2x$ or ② $y = 3 \sin 2(x + 45^o)$

7. (a)



(b) $f(x) \times g(x) < 0$ for $-90^o \leq x < -45^o$

8. (a) $a = 3$; $b = 2$.

(b) $3 \sin x = 2 \cos x$ or $\tan x = \frac{2}{3}$ [Divide both sides by $3 \cos x$] $\tan^{-1} {}^2/_3 = 33.7 + k\,180°$ with $k \in \mathbb{Z}$.
So the solutions are <u>$x = 33.7°$</u> or <u>$x = 213.7°$</u>

(c) for $0° \leq x < 33.7°$ or for <u>213.7°</u> $< x \leq 360°$

(d) $y = -2 \cos x$

<u>Section 10 Finding optimal solutions</u>

1. (a) $72 = x \times 2x \times h$ so $h = \dfrac{36}{x^2}$

(b) Total surface area $A = 2(x \times 2x + x \times h + 2x \times h) = 4x^2 + 2xh + 4xh = 4x^2 + 6xh = 4x^2 + \dfrac{216}{x}$ after substituting $h = \dfrac{36}{x^2}$.

(c) $\frac{dA}{dx} = 8x - \frac{216}{x^2} = 0$ or $8x^3 = 216 \rightarrow x^3 = 27$ so that $\underline{x = 3.}$

(d) $x = 3$ so $A = 36 + 216 \div 3 = 36 + 72 = \underline{108\ \text{cm}^3}$

Find $\frac{d^2A}{dx^2} = 8 + \frac{432}{x^3} > 0$ for all positive values of x so there is minimum value for $x = 3$ for A.

2. (a) Given: $480 = x \times 4 \times h \rightarrow h = \frac{120}{x}$

(b) Surface area of the box is: $A = 2(4x + 4h + xh) = 8x + h(8 + 2x) = 8x + \frac{120}{x}(8 + 2x) = 8x + \frac{960}{x} + 240$

(c) $\frac{dA}{dx} = 8 - \frac{960}{x^2} = 0$ or $x^2 = 960 \div 8 = 120$ so $x = \sqrt{120} \approx 11.0$. This is a minimum because $\frac{d^2A}{dx^2} > 0$

The value of the minimum area of cardboard used is: $8 \times \sqrt{120} + \frac{960}{\sqrt{120}} + 240 = \underline{415\ \text{cm}^2}$

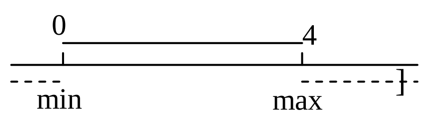
3. (a) $h'(t) = \frac{dh}{dt} = t - \frac{3}{4}t^2$ the value of $h'(3) = 3 - 0.75 \times 9 = -3.75$. So the depth is decreasing.

(b) $h'(t) = 0$ or $t - \frac{3}{4}t^2 = 0 \rightarrow t(1 - 0.75t) = 0$ so rate of water flowing into the tank, be the same as the rate of water flowing out of the tank at $t = 0$ (8:00 h) or at $t = 1 \div 0.75 = $ 1h 20 min so that is at 9:20 h

4. (a) $V = 4x^2(6 - x) = 24x^2 - 4x^3$.

(b) (i) $\frac{dV}{dx} = 48x - 12x^2 = 0$ or $12x(4 - x) = 0$ for $x = 0$ or for $x = 4$; $\frac{d^2V}{dx^2} = 48 - 24x$ so for $x = 4$ $\frac{d^2V}{dx^2} < 0$ so we have a maximum here.

[A sign diagram of $\frac{dV}{dx}$ indicates as well a maximum:



(ii) For $x = 4$, $V(4) = 24 \times 16 - 4 \times 64 = 64(6 - 4) = \underline{128\ \text{m}^3}$

5. (a) $\frac{dT}{dt} = T' = 2 - \frac{1}{2}t$

(b) **(i)** $T' = 2 - \frac{1}{2}t < 0$ for $-\frac{1}{2}t < -2$ multiply by -2: $t > 4$ so the temperature drops for $4 < t \leq 10$.

(ii) Since $0 \leq t \leq 10$ the temperature range is $T(4) - T(10) = 19 - 10 = \underline{9^\circ\ \text{C}}$.

6. (a) Express the TSA in terms of x and y: TSA = area 2 triangle + area 3 rectangles =

$2 \times \frac{1}{2} \times 3x \times 4x + y(3x + 4x + 5x) = 12x^2 + 12xy = 3600$ as is given.

Make y subject: $12xy = 3600 - 12x^2$ divide both sides by $12x$: $y = \frac{300 - x^2}{x}$

(b) First express the volume in terms of x: $V(x) = 6x^2 \times y = 1800x - 6x^3$. Then $\frac{dV}{dx} = 1800 - 18x^2$

$\frac{dV}{dx} = 0$ for $x = 10$ this is indeed a maximum value because $\frac{dV}{dx} > 0$ for $x = 9.5$ and $\frac{dV}{dx} < 0$ for $x = 10.5$.

7. **(a)** $\frac{dy}{dx} = 3ax^2 + 2bx + c$ At a turning point: $\frac{dx}{dy} = 0$ so $3ax^2+2bx+c = 0$…….*

Use the standard formula: $x = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$ for this case: $a_1 = 3a$; $b_1 = 2b$ and $c_1 = c$.

$$x = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1} = \frac{-2b \pm \sqrt{4b^2 - 4\times 3a \times c}}{6a} = \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$ qed.

(b) Apply: $x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} = \frac{-2 \pm \sqrt{4 - 3\times 1\times 10}}{3}$ since 4–30 < 0 there is no turning point.

8. **(a)** $B(t) = 1000 + 50t - 5t^2$ so $B'(t) = 50 - 10t$ and $B'(3) = 50 - 10 \times 3 = 20$.

(b) $B'(t) = 50 - 10t = 0$ for $t = 5$ h so 5h after adding the bactericide the no of bacteria will decrease.

(c) $B(t) = 0$ or $1000 + 50t - 5t^2 = 0$ or $t^2 - 10t - 200 = 0$ or $(t+10)(t-20) = 0 \rightarrow t = 20$.

So 20 hours after adding the bactericide all bacteria will be dead.

9. **(a)** Area cardboard used is $A = 4x^2 + 4xh$……*

(b) Volume is 1000 cm^3 so $x^2h = 1000$ or $h = 1000/x^2$. Substitute this in * and you get: $A = 4x^2 + \frac{4000}{x}$

(c) Differentiate A and equate to zero: $\frac{d}{dx}\left(4x^2 + \frac{4000}{x}\right) = 8x - \frac{4000}{x^2} = 0$ or $8x^3 = 4000$ so $x^3 = 500$ or

$x = \sqrt[3]{500} \approx 7.94$ cm and $h = 1000 \div x^2 = 15.9$ cm.

10. **(a)** $\frac{d}{dt}T = T' = -0.024t^2 - 0.16$**;** substitute $t = 5$: rate of change is $-0.024 \times 25 - 0.16 = -0.6 - 0.16 = -0.76$.

(b) $T(10) = 30 - 0.008 \times 10^3 - 0.16 \times 10 = 30 - 8 - 1.6 = 20.4$°C.

11. **(a)** Volume = 350 cm^3 = $\pi r^2 h$ so $h = \frac{350}{\pi r^2}$

(b) TSA = $A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{350}{\pi r^2} = 2\pi r^2 + \frac{700}{r}$

(c) $\frac{dA}{dr} = 4\pi r - 700r^{-2}$ for a minimum value $\frac{dA}{dr} = 0$ so $4\pi r - 700r^{-2} = 0$ or $4\pi r^3 = 700$ or $r = \sqrt[3]{\frac{700}{4\pi}} = 3.82$ cm.

<u>Section 11 Solving logarithmic equations</u>

1. $\log_2 x - 3\log_x 2 = 2.$ Use the change of base formula for logs: $\log_a b = \frac{\log b}{\log a}$

$\frac{\log x}{\log 2} - \log_x 8 = 2$ or $\frac{\log x}{\log 2} - \frac{\log 8}{\log x} = 2$ use the substitution: $\log x = p$

$\frac{p}{\log 2} - \frac{\log 8}{p} = 2$ multiply both sides by p: $\frac{p^2}{\log 2} - \log 8 = 2p$ or $\frac{p^2}{\log 2} - 2p - \log 8 = 0$

Use the formula to solve this quadratic equation: $p = \dfrac{2 \pm \sqrt{4 - 4\times\frac{1}{\log 2}\times(-\log 8)}}{\frac{2}{\log 2}} = \dfrac{2 \pm \sqrt{4+4\times\frac{3\log 2}{\log 2}}}{\frac{2}{\log 2}} =$

$= \dfrac{2 \pm 4}{\frac{2}{\log 2}} = (1 \pm 2)\log 2$ so $p = 3\log 2 = \log 8$ this means $\underline{x = 8}$ or $p = -\log 2 = \log ½$ this means $\underline{x = ½}$

[This is quite long better to go over to $\log_2 x$: $\log_2 x - 3\dfrac{\log_2 2}{\log_2 x} = 2.$ multiply with $\log_2 x \rightarrow$

$(\log_2 x)^2 - 2\log_2 x - 3 = 0$ so that $(\log_2 x - 3)(\log_2 x + 1) = 0$. Solutions are: $\log_2 x = 3 \rightarrow x = 8$ and $\log_2 x = -1 \rightarrow x = ½$.]

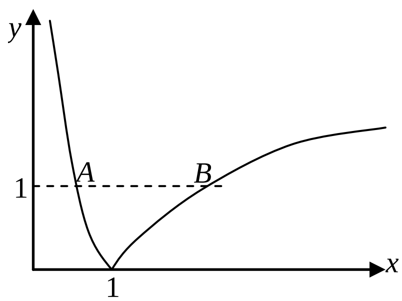
2. (a) Write $\log_x 10$ as a 10 based log: $\log_x 10 = \dfrac{\log 10}{\log x}$ so the equation becomes: $2\log x - \dfrac{\log 10}{\log x} = 1$

Multiply both sides with $\log x$: $2\log^2 x - \log x - 1 = 0$ [$\log 10 = 1$] For easy writing: $\log x = p$ then the equation changes to: $2p^2 - p - 1 = 0$ or $(2p + 1)(p - 1) = 0$ so that $p = 1$ or $p = -½$

$\log x = 1$ or $x = 10$ the other solution: $\log x = -½$ means $x = \frac{1}{\sqrt{10}}$.

(b) $y = ab^x \rightarrow$ take log: $\log y = \log a + x\log b \rightarrow \log b = 2$ so $\underline{b = 100}$ and $\log a = 3$ [the y intercept] so $\underline{a = 1000.}$

3.



4. (a) $x - 2 > 0$ so $x > 2$ and at the same time: $2x + 1 > 0$ or $x > -½$. So P only exists for $\underline{x > 2.}$

(b) $\log_3(x-2) + \log_3(2x+1) = 1$ or $\log_3(x-2)(2x+1) = \log_3 3$ so that $(x - 2)(2x + 1) = 3$ work out brackets:

$2x^2 - 3x - 2 - 3 = 0$ or $2x^2 - 3x - 5 = 0$ or $2x^2 - 5x + 2x - 5 = 0$ or $x(2x - 5) + 1(2x - 5) = 0$

$(2x - 5)(x + 1) = 0$ so that $\underline{x = 2½}$ [for $x = -1$, P does not exist see **(a)**]

5. $\log(12x + 4) > 1 + \log(x + 2) \rightarrow \log(12x + 4) - \log(x + 2) > \log 10$ or $\log\dfrac{12x+4}{x+2} > \log 10$ so

$\dfrac{12x+4}{x+2} > 10$ or $12x + 4 > 10(x + 2) \rightarrow 2x > 16$ so $\underline{x > 8}$ [for these values of x both logs exist and $x + 2 > 0$]

6. (a) $3\log p + \log_p 100 = 5.$ Use the 4th law of log to rewrite the second logarithm:

$3\log p + \dfrac{\log 100}{\log p} = 5.$ Let $\log p = q$ then the equation changes into: $3q + \dfrac{2}{q} = 5.$ Multiply with q:

$3q^2 - 5q + 2 = 0$; factorise: $(3q - 2)(q - 1) = 0$ so $q = 1$ with $p = 10$ or $q = \frac{2}{3}$ with $p = 4.64$

(b) Take log from both sides: $\log y = \log ax^b$ or $\log y = \log a + b\log x$ so $\underline{b = ½}$. Point (2, 5) is a point of the line, so $5 = \log a + ½ \times 2 \rightarrow \log a = 4$ then $\underline{a = 10000}$

7. Given: $3\log_8 x - 5 = 2\log_x 8.$

(a) $x > 0$ and $x \neq 1$

(b) $3\log_8 x - 5 = 2\log_x 8.$ Use the 4th law:

$3\log_8 x - 5 = 2\dfrac{\log_8 8}{\log_8 x}$ **or** $3\log_8 x - 5 = \dfrac{2}{\log_8 x}$ so let $\log_8 x = p$.

$3p - 5 = 2/p$ this changes into $3p^2 - 5p - 2 = 0$ factorize: $(3p + 1)(p - 2) = 0$ so

$p = 2$ then $\log_8 x = 2$ so $\underline{x = 64}$

$p = -\frac{1}{3}$ or $\log_8 x = -\frac{1}{3}$ then $\underline{x = ½}$

8. (a) Two methods are used: I let the base of the log remain ½; II go over to 2 as a base for the log.

Working	Comment
I $\log_{(\frac{1}{2})} x + \log_{(\frac{1}{2})}(x+1) \geq -1$ $\log_{(\frac{1}{2})} x(x+1) \geq \log_{(\frac{1}{2})} 2$ $x(x+1) \leq 2$ or $x^2 + x - 2 \leq 0$ or $(x+2)(x-1) \leq 0$ solution for $\underline{-2 \leq x \leq 1}$	Apply law 1 to write the left side as one log and write the right side as well in logarithmic form. Since the base no is less than 1 the sign changes. Factorize and conclude.

Working	Comment
II $\log_{(\frac{1}{2})} x + \log_{(\frac{1}{2})}(x+1) \geq -1$	Use law 4 to go over to base 2:
$\dfrac{\log_2 x}{\log_2 \frac{1}{2}} + \dfrac{\log_2 (x+1)}{\log_2 \frac{1}{2}} \geq \log_2 \frac{1}{2}$	But $\log_2 \frac{1}{2} = -1$ so the eq. can be rewritten:
$-\log_2 x - \log_2 (x+1) \geq \log_2 \frac{1}{2}$	Multiply both sides with -1; the sign changes!
$\log_2 x + \log_2 (x+1) \leq -\log_2 \frac{1}{2}$	But $-\log_2 \frac{1}{2} = \log_2 (\frac{1}{2})^{-1} = \log_2 2$
$\log_2 x(x+1) \leq \log_2 2$ so	From here on it works out the same as I.
$x(x+1) \leq 2$ or $x^2 + x - 2 \leq 0$ or $(x+2)(x-1) \leq 0$	
solution for $\underline{-2 \leq x \leq 1}$	

(b) (i) Use the formula: $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{9 \pm \sqrt{81 - 4\times1\times(-2)}}{2} = \dfrac{9 \pm \sqrt{89}}{2}$ which means $y = 9.11$ or $\underline{y = -0.22}$

(ii) Hence solve the equation $3^{2x} - 3^{x+2} - 2 = 0$ rewrite using $3^x = p$ we get $p^2 - 9p - 2 = 0$

So $3^x = 9.11$ take log: $x = \log 9.11 \div \log 3 = \underline{2.01}$ or $3^x = -0.22$ no solution. [A power of 3 cannot be negative]

9. (a) $3^{-2x} - 3^{2-x} + 18 = 0$ becomes $(3^{-x})^2 - 3^2 \times 3^{-x} + 18 = 0$; apply substitution $k = 3^{-x}$ and you get:

$k^2 - 9k + 18 = 0$ or $(k - 6)(k - 3) = 0$ so that $k = 3$ this means $\underline{x = -1}$ or $k = 6$ or

$3^{-x} = 6$ or $x = -\log 6 \div \log 3 = -1.63$

(b) $\log_3(x+1) + \log_{\frac{1}{3}}(2x-1) = 1$

use the change of base formula for the second term and the eq. changes into: $\log_3(x+1)+\frac{\log_3(2x-1)}{\log_3\frac{1}{3}}=1$

this can be rewritten as: $\log_3(x+1)-\log_3(2x-1)=1$ apply law 2: $\log_3\frac{x+1}{2x-1}=\log_3 3$ so

$\frac{x+1}{2x-1}=3$ or $x + 1 = 3(2x - 1)$ so that $\underline{x = 0.8}$

Section 12 Relation between variables

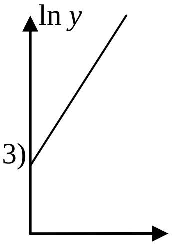
1 (a) Take the natural logarithm from both sides: $\ln y = \ln ae^{bx}$ or $\ln y = \ln a + bx$. ………..①
The graph of ln *y* against *x* is a straight line graph; with as independent variable *x* and dependant variable ln *y*.
The gradient is $\underline{b}$ and the ln *y* intercept: $\underline{\ln a}$.

(b) Line passes through (0, 0.6) and (2, 1.6) → replace (0, 0.6) in ①: $0.6 = \ln a$ so $\underline{a = e^{0.6} = 1.82}$.
replace (2, 1.6) in ①: $1.6 = 0.6 + 2b$ this means $\underline{b = ½}$.

(c) Substitute $x = 4$ in ① with *a* and *b* replaced by the values found in **(b):** $\ln y = 0.6 + 0.5x \rightarrow \ln y = 0.6 + 2 \rightarrow$
$\ln y = 2.6$ so $\underline{y = e^{2.6} = 13.5}$

2. (a) $y = 3e^{5x}$ take natural log form both sides: $\ln y = \ln 3 + 5x$
so $m = 5$ and $c = \ln 3$.

(b) (i) and (ii) see diagram.



(c) (i) $x = -1 \rightarrow y = 3e^{-5} = \underline{0.0202}$

(ii) $y = 90 \rightarrow 90 = 3e^{5x}$ or $e^{5x} = 30$ take natural log from both sides:
$5x = \ln 30$ so $x = (\ln 30) \div 5 = \underline{0.680}$.

3. (a) $x + 3 > 0 \rightarrow x > -3$ and $x - 3 > 0 \rightarrow x > 3$ so *y* exist for $\underline{x > 3}$.

(b) $4 = \log_2(x+3)(x-3)$ or $\log_2 16 = \log_2(x+3)(x-3)$ so $(x + 3)(x - 3) = 16 \rightarrow x^2 - 9 - 16 = 0$
$x^2 - 25 = 0 \rightarrow$ so $x = -5$ or $x = 5$ this means $\underline{x = 5}$ because $x > 3$.

(c) Answer: $x > 5$ both logs in $\log_2(x+3) + \log_2(x-3)$ are increasing functions.

4. (a) $R = ka^x$ take log from both sides: $\log R = \log ka^x$ or $\log R = x \log a + \log k$ …..*)
so $\log k = 0.6$ or $k = 4.0$ using the *y*-intercept of *).
Substitute (4, 0.4) in *) you get: $0.4 = 4 \log a + 0.6 \rightarrow 4 \log a = -0.20$ or $\log a = -0.05$ so $a = 10^{-0.05} = \underline{0.89}$

(b) Again substitute in *): $\log R = 10 \times(-0.05) + 0.6 = 0.1$ so $R = 1.26$.

Section 13 Functions, inverse and composite functions; completing the squares; quadratic theory

1. (a) $2x^2 + 8x - 10 = 2(x^2 + 4x - 5) = 2[x^2 + 4x + 4 - 4 - 5] = 2[(x^2 + 4x + 4) - 4 - 5] = 2[(x + 2)^2 - 9]$
After working out the square brackets you get: $2x^2 + 8x - 10 = 2(x + 2)^2 - 18$ so $a = 2$; $b = 2$ and $c = -18$.

(b) Least value of *y* is -18 for $x = -2$.

(c) $2x^2 + 8x - 10 = k \rightarrow 2x^2 + 8x - 10 - k = 0$ Discriminant > 0 or $b^2 - 4ac > 0$ substitute:
$64 - 4 \times 2 \times (-10 - k) > 0 \rightarrow 64 + 80 + 8k > 0 \rightarrow 8k > -144 \rightarrow \underline{k > -18}$.

(d) g is a one to one function; f is not. Only one to one functions have an inverse.

(e) $g(x) = 2x^2 + 8x - 10$ or $y = 2x^2 + 8x - 10$ or $y = 2(x + 2)^2 - 18$ swop *x* and *y*: $x = 2(y + 2)^2 - 18$
Make *y* subject of the equation: $2(y + 2)^2 = x + 18 \rightarrow (y + 2)^2 = ½x + 9 \rightarrow y+2 = \sqrt{\frac{1}{2}x+9}$

So $g^{-1}(x) = -2 + \sqrt{\frac{1}{2}x+9}$

(f) Domain is $½x + 9 > 0 \rightarrow ½x > -9 \rightarrow x \geq -18$ and the range is $y \geq -2$.

2. (a) fg(5) = f(g(5)) = f(ln 2) = $\dfrac{1-2\ln 2}{3+\ln 2}$ = <u>−0.105</u>

(b) ff(x) = f(f(x)) = $f\left(\dfrac{1-2x}{3+x}\right) = \dfrac{1-2\times\frac{1-2x}{3+x}}{3+\frac{1-2x}{3+x}} = \dfrac{\left(1-2\times\frac{1-2x}{3+x}\right)\times(3+x)}{\left(3+\frac{1-2x}{3+x}\right)\times(3+x)} = \dfrac{3+x-2(1-2x)}{9+3x+(1-2x)} = \dfrac{1+5x}{10+x}$

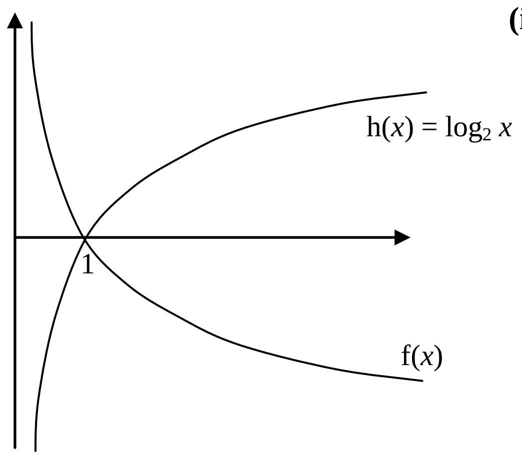
(c) f(0) = $\frac{1}{3}$ and g is only defined for $x > 3$ so gf(0) cannot be evaluated.

(d) $f^{-1}(x) \to y = \dfrac{1-2x}{3+x}$ swop x and y and make y subject: $x = \dfrac{1-2y}{3+y} \xrightarrow{\times(3+y)} x(3+y) = 1-2y$ rewrite:

$2y + xy = 1 - 3x \;\to\; y(2+x) = 1 - 3x \;\to\; f^{-1}(x) = \dfrac{1-3x}{2+x}$

$g^{-1}(x) \to y = \ln(x-3)$ write as a power: $e^y = x - 3$ swop the x and the y: $e^x = y - 3$ so $g^{-1}(x) = e^x + 3$

3. (a) (b) (i)



(ii) h(x) = $\log_2 x$

(c) $f(x) = -\log_2 x$ or $y = -\log_2 x$ or $y = \log_2 x^{-1}$ so $2^y = x^{-1}$ or $x = 2^{-y}$ so $f^{-1}(x) = 2^{-x}$
[Extra: Check: $f(f^{-1}(x)) = f(2^{-x}) = -\log_2(2^{-x}) = x$ indeed]

4. (a) range of f is $y \geq e$

(b) $y = e^{x+1}$ so $\ln y = x + 1$ or $x = \ln y - 1$ swop x and y : $y = \ln x - 1$ so $f^{-1}\, x \mapsto \ln x - 1$.

(c) fg(−4) = f(g(−4)) = f(7) = <u>e^8</u>

5. (a) fg(a) = 99 or f(g(a)) = f(a^2) = $4a^2 - 1 = 99 \to 4a^2 = 100 \to a^2 = 25$ so <u>$a = \pm 5$</u>

(b) first f^{-1}: $y = 4x - 1$ or $0.25y - 0.25 = x$ so $f^{-1}: x \mapsto 0.25x - 0.25$
So $gf^{-1}(b) = 81$ becomes $g(0.25b - 0.25) = 81$ or $(0.25b - 0.25)^2 = 81$ this means $0.25b - 0.25 = \pm 9$
Multiply with 4 and you get: $b - 1 = \pm 36$ so <u>$b = 37$ or $b = -35$</u>

(c) ff(c) = 27 becomes f($4x - 1$) = 27 or $4\times(4x-1) - 1 = 27 \to 16x - 5 = 27$ so that $16x = 32$ or <u>$x = 2$.</u>

(d) No g has no inverse because g is not a one to one function. Example g(x) = 25 has two solutions: ±5.

6. (a) $a - 3 > 0$ so $a > 3$. [Logarithm can only be taken from a positive number.]

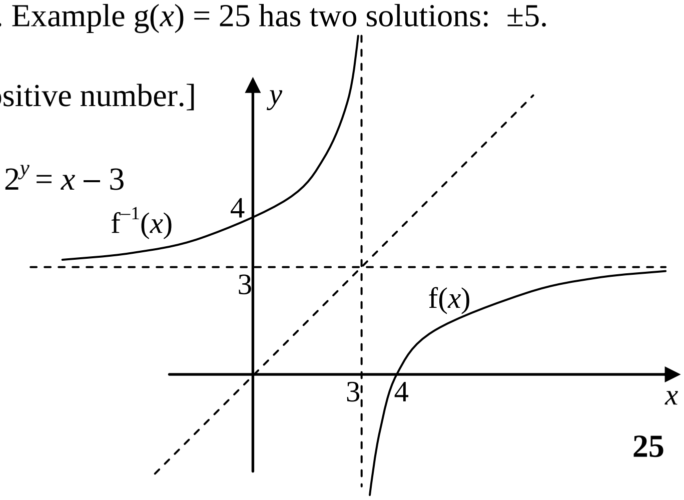
(b) (i) Range of f is ℝ

(ii) $f(x) = \log_2(x-3)$ or $y = \log_2(x-3)$; write as a power: $2^y = x - 3$
swop x and y: $2^x = y - 3$ so $y = 2^x + 3$
so $f^{-1}(x) = 2^x + 3$

(c) → → → → → → → →

(d) gf(7) = g($\log_2 4$) = g(2) = 1

(e) fg(1) = f(−1) = $\log_2(-4)$ cannot be done.



7. (a) fg(2) = f(ln 3) = $\dfrac{\ln 3 - 2}{\ln 3 + 3} = \dfrac{-0.901}{4.10} = -0.2$

(b) f(–1) = –3 ÷ 2 = –1.5 yet g only defined for $x > -1$.

(c) (i) $y = \dfrac{x-2}{x+3}$ swop x and y: $x = \dfrac{y-2}{y+3}$ or $x(y + 3) = y - 2$

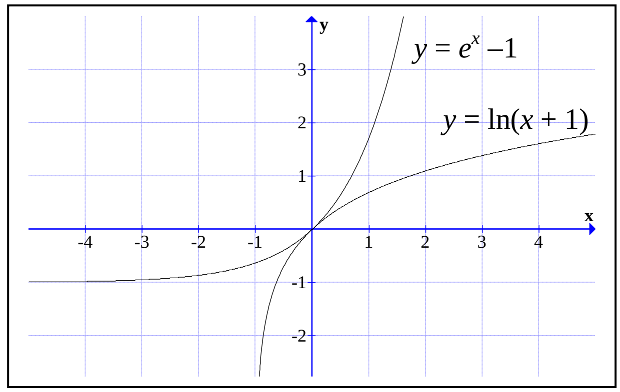
or $xy + 3x - y = -2$ or $y(x - 1) = -2 - 3x$

So that we get for $f^{-1}: x \mapsto \dfrac{-2-3x}{x-1}$

(ii) g: $x \mapsto \ln(x+1)$ for $x > -1$ or $y = \ln(x + 1)$ swop x and y and go over to exponential notation:

$x = \ln(y + 1)$ or $e^x = y + 1$ so that $y = e^x - 1$ and $g^{-1}: x \mapsto e^x - 1$

(d) See diagram.



8. (a) (i) Since $e^x > 0$ for all values of x; the range of f is $y > 3$

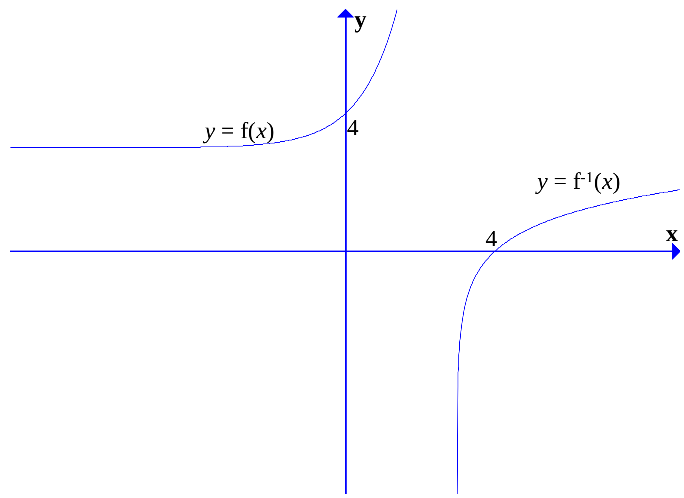
(ii) $y = e^x + 3$ so $x = e^y + 3 \rightarrow x - 3 = e^y$ take ln form both sides: $\ln(x - 3) = y$ so $f^{-1}(x) = \ln(x - 3)$

(iii) Domain of f^{-1} is $x > 3$.

(iv) See diagram to the right:

(b) (i) gf(4) = g(e^4 + 3) = 3(e^4 + 3) – 2 = 3 e^4 + 7 = 170.8

(ii) $f^{-1}g(-1) = f^{-1}(-5) = \ln(-8)$ this does not exist since the answer for any power of e is always positive.



Section 14 Sequences & patterns and series

1. $\sum_{r=0}^{23}(45 - 3r) = 45 + 42 + 39 + 36 + \ldots\ldots\ldots\ldots\ldots -21 -$

Or using the formula for an arithmetical sum (Take care there are 24 terms here):

$S_n = ½\, n\, [2a + (n - 1)d] = ½ \times 24\, [2 \times 45 + 23 \times (-3)] = 12 \times [90 - 69] = 12 \times 21 = \underline{252}$

2. (a) Decrease of 20% means the yield decreases with a factor 0.8. So the production is a geometrical sequence: 7000; 7000 × 0.8; 7000 × $(0.8)^2$; 7000 × $(0.8)^3$ …. This is a geometrical sequnce with a ratio < 1.

Sum is: $S = \dfrac{a}{1-r} = \dfrac{7000}{1-0.8} = \dfrac{7000}{\frac{1}{5}} = 35000$ kg of copper.

(b) $a = 7000$; $r = 0.8$ so $t_n = ar^{n-1} = 7000 \times 0.8^{n-1} < 1000$ first solve $7000 \times 0.8^{n-1} = 1000$ or $0.8^{n-1} = {}^1/_7$ take log from both sides: $(n - 1) \log 0.8 = \log {}^1/_7 \rightarrow n \log 0.8 = \log 0.8 + \log {}^1/_7$

$n = \dfrac{\log 0.8 + \log \frac{1}{7}}{\log 0.8} = 9.7$ years so the mine will be abandonded after <u>9 years</u>. [$T_9 = 7000 \times 0.8^8 = 1174$ kg]

But $T_{10} = 7000 \times 0.8^9 = 940$ kg] So in the 10^{th} year the production will fall below 1000 kg.

(c) $S_9 = \dfrac{a(1-r^9)}{1-r} = \dfrac{7000(1-0.8^9)}{1-0.8} = 35000\,(1-0.8^9) = 30300\,kg$

3. (a) (i) $t_1 = S_1 = 2 - 1 = \underline{1 = a}$

(ii) $S_2 = 6 = t_1 + t_2$ yet $t_1 = 1$ so $t_2 = 5$ so the difference $\underline{d = 4}$

(b) $T_3 = ar^2$ and $T_8 = ar^7$ then $T_8 \div T_3 = r^5$ yet $T_8 \div T_3 = \frac{1}{4} \div 8 = \frac{1}{32}$ so $\underline{r = ½.}$

(c) $\sum_{n=1}^{k} 5(3^n) = 442860$ or $5\sum_{n=1}^{k}(3^n) = 442860$ divide both sides by 5: $\sum_{n=1}^{k}(3^n) = 88572$ use the sum formula for gp:

$88572 = \dfrac{a(r^k - 1)}{r-1} = \dfrac{3(3^k - 1)}{3-1} \to 177144 = 3(3^k - 1)$ divide by 3: $59048 = 3^k - 1$ or $59049 = 3^k$

This equation can be solved with logaritm: $k = \dfrac{\log 59049}{\log 3} = \underline{10}$.

4. (a) $\sum_{r=1}^{20}(-2r+10) = 8 + 6 + 4 + \ldots\ldots\ldots - 28 - 30 = 10 \times (-22) = \underline{-220.}$

(b) (i) $t_2 = 16$.

(ii) So the sequence is like: 4, 16, 64, …..4^n use $S_n = \dfrac{a(r^n - 1)}{r-1}$ so $S_6 = \dfrac{4(4^6 - 1)}{4-1} = \dfrac{4 \times 4095}{3} = \underline{5460 \text{ letters.}}$

(c) $S = \dfrac{a}{1-r} = \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \dfrac{\frac{1}{3}\times 3}{\frac{2}{3}\times 3} = \dfrac{1}{2}$

5. (a) (i) Given $t_8 = 2t_4 \to a + 7d = 2(a + 3d)$ rewrite: $0 = a - d$ …….① and $t_{20} = 40$ so $t_{20} = a + 19d = 40$….②

② – ① gives: $20d = 40$ so $\underline{d = 2.}$ Using ① you get as well $a = 2$

(ii) Using ① you get as well $a = 2$ then $t_8 + t_9 + \ldots. t_{19} + t_{20} = S_{20} - S_7$. Use $S_n = ½\, n\, [2a + (n-1)d]$ so

$S_{20} - S_7 = \{½ \times 20[4 + 19 \times 2]\} - \{½ \times 7[4 + 6 \times 2]\} = 10 \times 42 - 56 = \underline{364}$

(b) (i) $\sum_{t=1}^{12}(24 - 3t) = 21 + 18 + \ldots. - 12 = 6 \times 9 = \underline{54.}$

(ii) $\underline{27p + 27p^2 + 27p^3}$

(iii) $S = \dfrac{a}{1-r} = \dfrac{27p}{1-p}$

(iv) $\dfrac{27p}{1-p} = 54$ or $27p = 54 - 54p$ so $81p = 54$ or $9p = 6$ so that $\underline{p = {}^2/_3.}$

6. (a) $t_5 = S_5 - S_4 = 5 \times 6 - 4 \times 5 = \underline{10}$

(b) $\sum_{k=4}^{\infty} 3^{-k+1} = 3^{-3} + 3^{-4} + \ldots = \dfrac{a}{1-r} = \dfrac{3^{-3}}{1-\frac{1}{3}} = \dfrac{1}{27} \div \dfrac{2}{3} = \frac{1}{27}\times\frac{3}{2} = \frac{1}{18}$

7. (a) (i) $T_1 = 4(m-1)^2$; $T_2 = 4(m-1)^3$ etc.so this is a geometric sequence with first term $4(m-1)^2$ and ratio $(m-1)$.

(ii) A series converges if the ratio r is: $-1 < r < 1$ or $-1 < m - 1 < 1$ add 1 to all three sides: $0 < m < 2$

The formula for the sum is: $S = \frac{a}{1-r}$ in which a is T_1 and r is the ratio. Substitute: $S = \frac{4(m-1)^2}{1-(m-1)} = \frac{4(m-1)^2}{2-m}$

(b) $\sum_{k=0}^{n} 5\times 3^k = 1328600$ this forms a geometric sequence with $a = 5$ and $r = 3$..

The sum formula of a geometric sequence is: $S_n = \frac{a(r^n - 1)}{r-1}$ substitute: $1328600 = \frac{5(3^n - 1)}{3-1}$ simplify:

$2 \times 1328600 \div 5 = 3^n - 1$ or $531441 = 3^n$ take log $\rightarrow n =$ log 531441 ÷ log 3 $\rightarrow$ <u>$n = 12$</u>

(c) Arithmetic progression then $x - 5 = y - x$ or $x = ½(5 + y)$….①

Geometric progression then $\frac{81}{y} = \frac{y}{x}$ cross multiplication: $81x = y^2$ ….②

Substitute ① into ②: $81 \times ½(5 + y) = y^2$ rewrite this quadratic equation to standard form:
$2y^2 - 81y - 405 = 0$ [Usually it takes time to find the proper number combination that factorizes the eq, so let us use the formula] $y = \frac{81 \pm \sqrt{81^2 + 3240}}{4} = \frac{81 \pm 99}{4}$ so $y = 45$ or $y = -4.5$.

$y = 45$ gives <u>$x = 25$</u> and $y = -4.5$ gives <u>$x = 0.25$</u>.

8. **(a)** $S_n = ½\, n[2a + (n-1)d] = 100a$ and for the last term: $t_n = a + (n-1)d = 9a$

Write $½\, n[2a + (n-1)d] = 100a$ as $½\, n[a + \underline{a + (n-1)d}] = 100a$ or substituting the underlined part:
$½n\,[a + 9a] = 100a$ or $½n \times 10a = 100a$ divide by $10a$: $½n = 10$ so <u>$n = 20$.</u>

(b) **(i)** $T_n = 3(m-1)^{n+1}$ so $T_1 = 3(m-1)^2$; $T_2 = 3(m-1)^3$ and $T_3 = 3(m-1)^4$ and so on.

So this is a geometric sequence with first term: $a = 3(m-1)^2$ and $r = (m-1)$.

(ii) For $-1 < r < 1$ or $-1 < m - 1 < 1$ add 1: $0 < m < 2$ the sum to infinity is $S = a \div (1 - r) =$

$= a \div [1 - (m-1)] = a \div [m-2] = \frac{3(m-1)^2}{m-2}$

9. [Paper 2 2015 Q7]

(a) So $a = 0.4$ and $r = 0.8$ then $S = \frac{a}{1-r} = \frac{0.4}{0.2} = \frac{4}{2} = 2$

(b) $a = -3$ and $d = 6$ so $t_n = a + (n-1)d = -3 + 6(n-1) > 500$ or $6(n-1) > 503$ or $n - 1 > 83.83$ so <u>$n = 85$</u>

(c) $\sum_{k=1}^{n} 2\times 3^k = 43046718$. So $6 + 18 + 54 + \ldots\; 2 \times 3^n = 43046718$ this is a GP with $a = 6$ and $r = 3$.

The sum formula is $S_n = \frac{a(r^n - 1)}{r-1} = \frac{6(3^n - 1)}{2} = 43046718$ or $3^n - 1 = 14348906$ so that $3^n = 14348907$

Use logs to solve this exponential equation: n = log 14348907 ÷ log 3 = <u>15.</u>

10. **(a)** **(i)** $T_{20} = ar^{19} = 300 \times \left(\frac{15}{16}\right)^{19} =$ <u>88 cm</u>

(ii) $S_{20} = \frac{a(1-r^n)}{1-r} = \frac{300[1-\left(\frac{15}{16}\right)^{20}]}{1-\frac{15}{16}} = 16\times 300\times[1-\left(\frac{15}{16}\right)^{20}] = 34.8$ m

(b) There are 20 terms in 11, 14, 17, …, 68. [One can see this by subtracting 8 from each term.]

So <u>$x = 1$; $a = 3$ and $b = 8$.</u>

(c) $t_1 = m$ and $t_2 = 6$ so $r = \frac{6}{m}$ and $S = \frac{a}{1-r} = 25 = \frac{m}{1-\frac{6}{m}}$ use cross multiplication: $25 - \frac{150}{m} = m$; multiply with m:

$25m - 150 = m^2$ or $m^2 - 25m + 150 = 0$ or $(m-10)(m-15) = 0$ so <u>$m = 10$</u> or <u>$m = 15$.</u>

Section 15 Inequalities; algebraic fractions and completion of the square.

1. $x^2 - 2x - 2 > 1$ or $x^2 - 2x - 3 > 0$ first solve $x^2 - 2x - 3 = 0 \rightarrow (x - 3)(x + 1) = 0$ so $x = -1$ or $x = 3$

The solutions for $x^2 - 2x - 2 > 1$ are now: $x < -1$ or $x > 3$, because we are dealing here with a ∪ parabola.

2. (a) $2x^2 + x - 15 < 0$ first solve $2x^2 + x - 15 = 0$ or $2x^2 + 6x - 5x - 15 = 0$ or $2x(x + 3) - 5(x + 3) = 0$ so that

$(x + 3)(2x - 5) = 0$ so $2x^2 + x - 15 < 0$ for $-3 < x < 2.5$ because we are dealing here with a ∪ parabola.

(b) **(i)** $y = -2x^2 - 4x + 1 = -2(x^2 + 2x + 1 - 1) + 1 =$
$-2(x^2 + 2x + 1) + 2 + 1 = -2(x + 1)^2 + 3$ so $A = -2$; $B = 1$ and $C = 3$.

(ii) Range $\underline{y \le 3.}$

3. (a) **(i)** $y = -x^2 - 4x + 1 = -1[x^2 + 4x + 4 - 4] + 1 = -1[x^2 + 4x + 4] + 4 + 1 = -1(x + 2)^2 + 5$.
$\underline{\text{So } a = -1;\ b = 2 \text{ and } c = 5}$

(ii) Turning point is $\underline{(-2, 5)}$

(iii) The coefficient of x^2 is negative so this turning point is a $\underline{\text{maximum.}}$

(b) **(i)** $f(-5) = -4$ and $f(0) = 1$ so the range is $\underline{-4 \le y \le 5}$ [The x coordinate of the turning point lays within the given interval!]

(ii) f^{-1} does not exist because f is not a one to one function on the given domain.

4. (a) $$\frac{A}{x+2} - \frac{B}{x-3} = \frac{A(x-3) - B(x+2)}{(x-3)(x+2)} = \frac{x(A-B) - 3A - 2B}{(x-3)(x+2)}$$

(b) From **(a)** $A - B = -2$ …..① and $-3A - 2B = -9$……② then $2 \times$ ① – ② gives:

$$\begin{array}{rl} & 2A - 2B = -4 \\ - & -3A - 2B = -9 \\ \hline & 5A \quad = \quad 5 \end{array}$$

So $\underline{A = 1}$ and by substituting $A = 1$ in ① you get: $\underline{B = 3}$

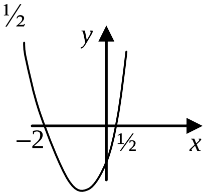
5. **(i)** $y^2 - 4y = 1$ or $y^2 - 4y - 1 = 0$ use the formula: $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$.

The '+' will give $y = 4.24$ and the '–' will give $y = -0.236$

(ii) $2^{2x} - 2^{x+2} = 1$….①, let $p = 2^x$ then ① becomes: $p^2 - 4p - 1 = 0$

Using the positive solution from **(i)** we get: $2^x = 2 + \sqrt{5}$ take logarithm: $x = \dfrac{\log(2+\sqrt{5})}{\log 2} = \underline{2.08}$

6. (a) $2x^2 + 3x \ge 2$ rewrite: $2x^2 + 3x - 2 \ge 0$ factorize: $(2x - 1)(x + 2) \ge 0$ so for $x \le -2$ or $x \ge ½$



(b) **(i)** $-2x^2 - 4x + 1 = -2(\underline{x^2 + 2x + 1} - 1) + 1 = -2\{\underline{(x + 1)^2} - 1\} + 1 = -2(x + 1)^2 + 3$.

So $\underline{a = -2;\ B = 1 \text{ and } C = 3}$.

(ii) Turning point of $y = -2x^2 - 4x + 1$ is $(-1; 3)$

7. Algebraic solution by equating the two equations: $cx + 4 = x^2 - 2x + 20$

Rewrite to standard form: $x^2 - 2x - cx - 4 + 20 = 0$ or $x^2 - (2 + c)x + 16 = 0$.

No solutions or $D < 0$ or $b^2 - 4ac < 0$ or $(2 + c)^2 - 4 \times 1 \times 16 < 0$ or $4 + 4c + c^2 - 64 < 0$

Rewrite in standard form: $c^2 + 4c - 60 < 0$ factorize: $(c + 10)(c - 6) < 0$.

No intersection with the curve for: $-10 < c < 6$.

Using calculus: Find the derivative: $\dfrac{dx}{dy} = 2x - 2$

In general the point with x-coordinate a, has a tangent with gradient $2a - 2$. This point has y-coordinate $y = a^2 - 2a + 20$. So the point of touch is $(a, a^2 - 2a + 20)$ and the eq. of the tangent $y = (2a - 2)x + c'$
$(a, a^2 - 2a + 20)$ is a point on this line: $a^2 - 2a + 20 = (2a - 2)a + c'$
Make c' subject: $c' = -a^2 + 20$ and $c' = 4$ so $-a^2 + 20 = 4$ or $a = \pm 4$.
So the gradients of the line of touch to the parabola are $a = 4 \rightarrow m = 6$ and $a = -4$ $m = -10$
So the line will not intersect with the curve for $-10 < c < 6$. [This last conclusion can only be drawn realizing the turning point is (1, 19) and the line is passing through (0, 4).If you cannot see this, sketch the parabola]

8. (a) (i) $y = 2x^2 - 12x + 16 = 2[x^2 - 6x] + 16 = 2[(x - 3)^2 - 9] + 16 = 2(x - 3)^2 - 2$. So $\underline{a = 2;\ b = -3 \text{ and } c = -2}$.
(ii) So the turning point is (3, –2)
(iii) $a > 0$ so it is a minimum. Or check $\frac{d^2y}{dx^2} = 4$ which is positive. So it is a minimum.
(b) (i) f(5) = 50 – 10 + 16 = 56. So the range is $-2 \le y \le 56$
(ii) f^{-1} does not exist because f(4) = f(2) = 0. So f^{-1} maps 0 on 2 and 4.

9. (a) $2x^2 - 13x < -15$ or $2x^2 - 13x + 15 < 0$ or $(2x - 3)(x - 5) < 0$. Now think of the graph of $y = (2x - 3)(x - 5)$; it has zeros in $x = 1\frac{1}{2}$ and $x = 5$ so $2x^2 - 13x < -15$ for $\underline{1\frac{1}{2} < x < 5}$.
(b) (i) The expression $-2x^2 + 12x - 13$ is identical to $a(x + p)^2 + q = ax^2 + 2axp + ap^2 + q$.
This means $\underline{a = -2}$ [Both expressions are identical so the coefficient in front of x^2 are the same.]
As well: $2ap = 12$ or $-2p = 6$ or $\underline{p = -3}$. Finally $-13 = ap^2 + q$ or $-13 = (-2) \times (-3)^2 + q$ or $\underline{q = 5}$.
(ii) The expression can be written as $-2(x - 3)^2 + 5$ so the turning point is $\underline{(3; 5)}$.